BRITISH ASSOCIATION

MEETING AT GLASGOW 1901

DISCUSSION ON THE

# TEACHING OF MATHEMATICS

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# DISCUSSION ON THE
# TEACHING OF MATHEMATICS

Which took place on September 14th, at a Joint Meeting of two Sections

Section A.—Mathematics and Physics
Section L.—Education

CHAIRMAN OF THE JOINT MEETING
THE RIGHT HON. SIR JOHN E. GORST, K.C., M.P.
*President of Section L*

EDITED BY PROFESSOR PERRY

To which is now added the Report of the British Association Committee drawn up by the Chairman Professor FORSYTH

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*Second Edition*, 1902

# PROCEEDINGS

Some men who intended to speak in the discussion were told by me in July that the Committee of Section L had arranged for it to take place on Monday, the 16th of September. But it was found that a change of date was imperative, and it was finally settled on the 13th that the discussion would take place next day, so that I had no opportunity of communicating with intending speakers who had not yet arrived in Glasgow.

The proceedings began with my address. A few copies of proofs of a small portion of the address reached me on the previous day, and although I was able to distribute complete copies just when they arrived from London before the beginning of the meeting, the speakers cannot be said to have had a fair opportunity of studying the address or the syllabus. The proofs were uncorrected by me. I read hurriedly, skipping much, and did not read the syllabus. The address was written in scraps during my holidays, and there had been no time to get it into a finished shape. I have no right now to correct even obvious verbal errors, as some of the remarks of some of the speakers refer to these errors.

The reporters' transcripts were in some cases very complete, in others very incomplete: they were sent to speakers with the request that they might be edited, and that additions might be

made in square brackets. I asked also for suggestions of names of a few other interested people to whom I might write asking for remarks. I have taken the liberty of writing a Reply.

The time ot every speaker in the discussion was limited to ten minutes. The following list gives in order the names of the speakers. I also give the names of people who have been kind enough to communicate remarks. Some of these remarks arrived too late to be referred to in my Reply.

After the discussion, at the request of Sections A and L, the general committee of the British Association appointed a committee, whose composition and functions are stated below. It is hoped that the publication ot this discussion will help the committee in its work, although, of course, nothing here stated can affect the committee's freedom of action.

JOHN PERRY.

*December*, 1901.

In this reprint I venture to include a communication made to the British Association Committee by twenty-two masters of public schools, and also the report of the committee drawn up by the chairman.

J. P.

*September*, 1902.

## SPEAKERS

PROFESSOR HUDSON, M.A. King's College, London.

PROFESSOR FORSYTH, M.A., D.Sc., F.R.S. Cambridge.

MAJOR PERCY MACMAHON, R.A., D.Sc., F.R.S., President of Section A.

E. M. LANGLEY, ESQ., M.A. Bedford Modern School.

PROFESSOR EVERETT, M.A., D.C.L., F.R.S. Late of Queen's College, Belfast.

PRINCIPAL RÜCKER, M.A., D.Sc., SEC.R.S. The University of London.

PROFESSOR SILVANUS P. THOMPSON, D.Sc., F.R.S. Finsbury Technical College.

MRS. W. N. SHAW.

PROFESSOR GREENHILL, M.A., F.R.S. Royal Artillery College, Woolwich.

JAMES PARKER SMITH, ESQ., M.P.

MRS. NATHANIEL LOUIS COHEN.

PROFESSOR ALFRED LODGE, M.A. Royal Indian Engineering College, Cooper's Hill.

PROFESSOR L. C. MIALL, F.R.S. The Yorkshire College, Leeds.

PROFESSOR MINCHIN, M.A., F.R.S. Royal Indian Engineering College, Cooper's Hill.

ANDREW JAMIESON, ESQ., M.INST.C.E., F.R.S.E. Emeritus Professor of Electrical Engineering, the Glasgow and West of Scotland Technical College.

WILLIAM R. COOPER, ESQ., M.A., B.Sc.

THE RT. HON. SIR JOHN E. GORST, K.C., M.P., President of Section L.

## AUTHORS OF WRITTEN REMARKS

PRINCIPAL OLIVER LODGE, D.SC., LL.D., F.R.S. The University of Birmingham.

OLIVER HEAVISIDE, ESQ., F.R.S.

THE RT. HON. THE LORD KELVIN, G.C.V.O., F.R.S.

JOSEPH LARMOR, ESQ., D.SC., F.R.S. St. John's College, Cambridge.

W. N. SHAW, ESQ., M.A., F.R.S. The Meteorological Office.

W. P. WORKMAN, ESQ., M.A., B.Sc. Kingswood School, Bath.

W E. SUMPNER, ESQ., D.Sc. Municipal Technical College, Birmingham.

W. D. EGGAR, ESQ., M.A. Eton College.

A. J. PRESSLAND, ESQ., M.A., F.R.S.E. Edinburgh Academy.

SIR PHILIP MAGNUS, B.Sc. City and Guilds of London Institute.

MRS. MARV EVEREST BOOLE.

MISS CHARLOTTE ANGAS SCOTT. Bryn Mawr College, Pennsylvania, U.S.A.

PROFESSOR DAVID EUGENE SMITH. Teachers College, Columbia University, New York City.

PROFESSOR HORACE LAMB, F.R.S. Owens College, Manchester.

G. B. MATHEWS, F.R.S. Formerly Professor of Mathematics in the University College of North Wales.

Twenty-two Masters of Public Schools.

# BRITISH ASSOCIATION COMMITTEE

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PROFESSOR A. R. FORSYTH.

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To report upon improvements that might be effected in the teaching of Mathematics, in the first instance in the teaching of Elementary Mathematics, and upon such means as they think likely to effect such improvements.

THE

# TEACHING OF MATHEMATICS

By Professor John Perry, M.E., D.Sc., LL.D., F.R.S.

For many years mathematicians have occasionally denounced our methods of teaching mathematics from their own point of view. I might quote Professor Sylvester, Professor De Morgan, and many others. Some of my friends who are interested in the education of possible mathematicians, and others like myself who are more interested in the education of the average citizen, feel that we ought not to wait longer in obtaining from such an audience as is here present an authoritative statement on this subject.

I beg to present to you a specimen syllabus which I have recently prepared for training colleges; any syllabus following more orthodox lines may be adopted by any training college, but as a member of a departmental committee which has recently been sitting, I have recommended this one.

It is to be remembered that courses of instruction adopted in training colleges are very likely to be adopted in primary schools, in continuation schools, and in many secondary schools. I have been allowed by the Science and Art Department to introduce this method of mathematical teaching in many evening science-schools and technical colleges. In some technical schools the usual pure mathematics syllabus has already been discarded altogether in favour of the new one. It is obvious, therefore, that what I am doing may have far-reaching consequences, and I beg, gentlemen, that you who represent so well every kind of authority on this subject will give me the benefit of your severest criticism and advice.

This is not to be a mere academic discussion. Anybody who thinks that I am making a mistake, or who sees how my method may be improved and who holds his tongue, is doing a real harm to the country. In a few years this new system will have become to some extent crystallised, and it will not then be so easy as it is now to get changes made in it; it will be a very difficult thing indeed to get it discarded altogether. I have shown that I recognise the greatness of my responsibility, but I hope that you will see that you have a duty to perform.

I have taught mathematics and applied science or engineering to almost every kind of boy or man. I have had my present notions almost as far back as thirty years ago. When I was young I felt diffident: now I have lost most of that feeling, because all my experience has confirmed my opinions and has shown me that many other teachers have the same views as myself.

I am about to say again to you what I said in 1880, in a paper read before the Society of Arts, and what I have said many times since then. If I could think that you had read the book containing some of my papers, of which I have sent a copy to almost all of you, it would not be necessary for me to say much now.

In presenting the syllabus, I want it to be clearly understood that I recommend this method, not only for classes of engineer apprentices, not only for children in Board schools, not only for the average British boy, but for the boys of very acute intellect who form less than one per cent. of the higher-class population, as well as for the few boys—say one in ten millions—who are likely to become great mathematicians.

I have said that it is usefulness which must determine what subjects ought to be taught to children and in what ways, and as I have been blamed for this, I should like to say something about the *utility* of the study of mathematics.

Although we can understand the old "mark-time" philosophers, who loved to plough the sand for the thousandth time and never reaped any harvest, saying with Seneca, "Invention of useful things is drudgery for the lowest slaves," surely it

can only be affectation in a mathematician of the twentieth century to echo, "It is an affront to geometry to say that it led to the principle of the arch," or "the sole function of astronomy is to assist in raising the mind to the contemplation of things which are to be perceived by the pure intellect alone." In truth, the pure mathematician is just like the rest of us. People outside any study are apt to think the student foolish if his results are not useful to outsiders. The student, irritated that an ignorant outsider should apply a stupid standard to his work, is tempted to say that in so far as he is useful to outsiders he is hateful to himself. But this is temporary irritation. The pure mathematician is pleased when his discoveries are of use to the physicist. The physicist is pleased when his discoveries are of use to the engineer. And, whether they like it or not, it is true that the engineer does often suggest new departures in physics and the physicist does often suggest new departures to the pure mathematician.

What an affectation it is for a student to say that his study is *useless!* The pursuit of pure knowledge for its own sake is one of the noblest of occupations. But why? Surely because of its many-sided usefulness, one side being its development of the mental power and soul, and indeed one may say the emotions of the student. And because a worker in some quite different branch of science greatly ignorant of the first branch gets from a discovery in the first branch an idea which helps him in his own work, surely, surely

> "Nothing is here for tears, nothing to wail
> Or knock the breast. . . ."

Let not, then, the pure mathematician be angry with me if I, an outsider, hold the view that my study is nobler than his. Let him keep to his ideals; but to me the pure mathematician is great because he has helped me so much in what I consider important things. The politician has his ideals, and so has the financier; but to me their value consists in what they have done to help forward what seem to me important things. What we want is a great Toleration Act which will allow us all to pursue our own ideals, taking each from the other what he can in the way of mental help. We do not want to interfere with the

students of pure mathematics, men whose peculiar mental processes are suited to these studies, men whose labours cannot be spared from the world's service. We believe that, the more they hold themselves in their studies as a race of demigods apart, the better it may be for the world. They are pursuing a branch of science for its own sake, and doing it all the better for the belief they hold that pure mathematics ought to be studied with no view to its application. But all the same I hold that the study began because it was useful, it continues because it is useful, and it is valuable to the world because of the usefulness of its results. The pure mathematician must allow me to go on thinking that, if his discoveries were not being utilised continually, his study would long ago have degenerated into something like what the Aristotelian dialectic became in the fourteenth century.

I belong to a great body of men who apply the principles of mathematics in physical science and engineering; I belong to the very much greater body of men who may be called persons of average intelligence. In each of these capacities I need mental training and also mathematical knowledge. The mathematician says that he wants to have nothing to do with us; but it is too late to say this sort of thing. It is he who has fixed how his subject shall be taught to us in schools, and he provides us with teachers of it. We pay these teachers to give us something that will be useful in our education and useful to us in life, useful to us in understanding our position in the universe. Surely we have a right to ask the mathematicians to look at this matter from our point of view, and to ask if it is not possible to help us without hurting themselves and their study. Without the help of the mathematicians I feel that it is almost impossible to get the teachers of mathematics to give us instruction of a useful kind.

I have hurriedly put together what strike me as obvious forms of usefulness in the study of mathematics.

(1) In producing the higher emotions and giving mental pleasure. Hitherto neglected in teaching almost all boys.

(2) *a* In brain development. *b*: In producing logical ways of thinking. Hitherto neglected in teaching most boys.

(3) In the aid given by mathematical weapons in the study of physical science. Hitherto neglected in teaching almost all boys.

(4) In passing examinations. The only form that has not been neglected. The only form really recognised by teachers.

(5) In giving men mental tools as easy to use as their legs or arms; enabling them to go on with their education (development of their souls and brains) throughout their lives, utilising for this purpose all their experience. This is exactly analogous with the power to educate one's self through the fondness for reading.

(6) Perhaps included in (5): in teaching a man the importance of thinking things out for himself and so delivering him from the present dreadful yoke of authority, and convincing him that, whether he obeys or commands others, he is one of the highest of beings. This is usually left to other than mathematical studies.

(7) In making men in any profession of applied science feel that they know the principles on which it is founded and according to which it is being developed.

(8) In giving to acute philosophical minds a logical counsel of perfection altogether charming and satisfying, and so preventing their attempting to develop any philosophical subject from the purely abstract point of view, because the absurdity of such an attempt has become obvious.

I believe that all these functions would be performed well under the new system which is suggested. At present, with the exception of (4), which is not particularly nice, in the performance of these functions mathematics affects less than one per cent. of the boys who are supposed to study it. I am aware that it is the taunt of some mathematical teachers that they are the caretakers of a useless and worshipful holy of holies, but I would have them consider if they have ever themselves been anywhere but in the outer courts of the temple where the shrines are tawdry and the traffic in sacrificial animals is nauseous. With the great leaders of mathematical thought here present I thought I could have no quarrel. I hope that they will agree with me when they understand my scheme. I know that I have the sympathy of many of them, and that they will forgive what might seem to some people my impudence in speaking about mathematics before them. The very severe remarks of the President of Section A have been reported at full length in the newspapers. I

am sorry to think that I have had so little success in explaining my proposed reform. I am inclined to think that, when I have explained it, he will really approve of what he seems now to condemn. He used a very happy illustration of the delights of the specialist in pure mathematics—that waterfall in Labrador, finer than Niagara, which had only been seen by nine white men; and I thought there was more than a little selfishness in his taking that view of matters. Using the illustration in explaining my reform, may I say that by building a railway in Labrador, by making bridges and paths, and hanging wire ropes, and cutting steps, I hope to throw that beautiful scene of which he spoke open to the gaze of many thousands of people. Surely he will see that it was not created for the enjoyment of nine men, however white. And I am afraid, very much afraid, that, in spite of him, engineers will proceed to utilise the power of that waterfall.

Fifty years ago it was thought right to teach physics and chemistry to all men in the same way; as if all men were to be physicists or chemists. The great growth of these subjects made a change necessary, and the newness of the studies made a change possible. For the same reasons I would teach mathematics, at all events advanced mathematics, in quite different ways to different students. In any case, I feel sure that our system of teaching boys elementary mathematics as if they were all going to be pure mathematicians must be altered. Perhaps the mathematicians will forgive my impertinence in saying that even for the boy who is likely to become a great mathematician I advocate improved methods of teaching: I say that the old Alexandrian method is bad. It is immensely important that if any one method of elementary teaching be generally adopted it shall not be hurtful to the one boy in a thousand who is fond of abstract reasoning, but it is just as important that the average boy shall not be hurt. In the heroic times every traveller was asked an enigma; if he did not answer he was killed with torments; if he answered, he was declared a demigod and given to rule over nations. In those times it was thought good to sacrifice myriads of people for the purpose of finding the one demigod.

So now we teach all boys what is called mathematical philosophy, that we may catch in our net the one demigod, the one pure mathematician, and we do our best to ruin all the others.

It is Nature's way with fishes: 10,000 herrings spawned for one survivor; 10,000 salmon eggs for one marketable fish; 10,000 Toms, Dicks, and Harrys mentally destroyed for the sake of producing one man fit to be a mathematical master of a second-rate public school; 10 million destroyed for the sake of producing one great mathematician.

But I would like to point out that ours is an altogether foolish way of producing the mathematical master also. To the age of twenty-four one may say that the very brightest mathematical minds of the world are being trained on problems about which there is nothing new, in the study of which there is absolutely no chance of a new discovery. The apparatus of this gymnasium of ours differs very little now from what it was thirty or fifty or eighty years ago. One cannot help referring to the similar state of things in regard to the scholastic philosophy. Roscelin, Anselm, Abelard, Peter Lombard, Albertus Magnus, Thomas Aquinas, Duns Scotus, Ockham, were the mental giants of their own time; they were men of the most acute and profound understanding. And the pupils of all their followers traversed the Aristotelian ground in exactly the same way, one generation after another. No advance made by a master was utilised in teaching his pupil. John of Salisbury observed of the Paris dialecticians of his own time that after several years' absence he found them not a step advanced and still employed in urging and parrying the same arguments. Bring up students to the age of twenty-four in such a system, and how few of them will show that they can get out of it ever after. Let a young mathematician know that all his difficulties were solved long ago; that almost no old mathematical man he meets ever makes a new discovery, nor wants to make one, and you give him the mental attitude of the Schoolmen.

Consider the really clever mathematical men whose academic education is completed in any particular year. Shall we say that one of them becomes a mathematician? that is, a man who extends the boundary of mathematical knowledge. And what of the others, of those whose education has ceased? I think you must admit that if they do not utilise their mathematical knowledge in the study of physical science, their mathematical horizon gets smaller and smaller. From the mathematical point of view they become vegetables. They are not even like the Greek

scholars of Constantinople during the dry-rot 1000 years period, when the world was handed over to the scholastic devil for 1000 years. The students in Constantinople really read and copied manuscripts and took an interest in learning, but the ordinary mathematical man, as he vegetates, seems to take no interest in mathematics, and never opens a mathematical book except to keep him right when he is drilling the young recruit. Observe that I say nothing about him as a citizen, as an authority on the training of the mind, as a centre of intellectual light for the people among whom he dwells; I speak of him as a mathematician merely.

Now in my experience there is hardly any man who may not become a discoverer, an advancer of knowledge, and the earlier the age at which you give him chances of exercising his individuality the better. Put him in command of all [1] existing knowledge as quickly as possible, and while doing it let him know that he also has the marshal's *baton* in his knapsack. Let him know that he is expected to be making discoveries all the time; not merely that the best established law is not complete, but that in the very simplest things it is not so much what he is told by a teacher, but what he discovers for himself, that is of real value to him, that becomes permanently part of his mental machinery. Educate through the experience already possessed by a boy; look at things from his point of view—that is, lead him to educate himself. I feel that throughout one's whole mathematical course it is important to teach a student through his own experiments, through concrete examples worked out by him. Even good mathematical teachers hate to see their students "wasting time," as they call it, in actually plotting lines of force or stream lines after the algebraic academic answer has been arrived at. They would probably call it waste of time to caulk the joints of plates in a ship. Without this kind of illustration I feel sure that the whole study is useless except for one man in a thousand. I am here speaking of advanced work. But even in elementary work a student ought to be induced to apply his mathematics to problems in his own experience.

When I was an apprentice I knew some trigonometry, much

[1] In a corrected proof this word *all* would have been altered to *much essential*.

more algebra, and I was an adept in geometry. I wanted to know engineering theory, and the only two books available contained unknown mathematical symbols. Other boys might sigh for other luxuries, but to me the one thing wanting was a knowledge of

$$\frac{dy}{dx} \text{ and } \int.$$

Looking back, I seem to have panted for a knowledge of the use of these symbols for years. There was nobody to give me advice: I knew many clever engineers, but they could not help me. At length I had an opportunity of getting this knowledge. Now I was as well prepared as my fellow students, and yet somehow I lagged behind in exercise work. They dashed easily through twenty examples at the end of a chapter; to me every example was a labour, an interesting labour; but truly a difficult job. And all through that college session I was filled with a sense of my stupidity; satisfied with my progress in itself, but utterly dissatisfied when I compared myself with the others.

Now I know wherein the difference consisted. I was really using the idea of the calculus in all sorts of problems outside the academic ones; making them part of my mental machinery. They had the knack (their previous training had all been in the direction of giving them this knack) of rapidly picking up just such instruction as enabled them to do the examples, and for them there was nothing else.

If I had time I could illustrate my meaning in many ways. From my observation of men and boys I am inclined to think that my way of taking up a study is the common way, the natural way, and that the schoolmasters destroy it and replace it by something that conduces to mere learning.

In this connection it is worth remarking that if a student has been learning and not discovering, when at length as a man he is so exceptional as not to have become *stale* and he does make a discovery, he thinks too much of this child of his old age; he squabbles in the most vulgar manner for priority of discovery. If he had been discovering things all his life he would let anybody claim priority who cared to do so. The fame of having made a discovery would be no great reward for a man from whom

an inward glow of pioneering satisfaction had never been absent since he began his studies.

I do not care here to dwell on the fact that, if a student has been increasing his learning all his life, he will make too much of the importance of mere learning.

The unpractical character of mathematical teaching causes mathematical men to leave common-sense out of all teaching. Good illustrations of this come from Germany, France, and Switzerland. An engineer needs a knowledge of graphical statics. A good practical course of a few weeks will give him such a thorough grasp of the general principle as will enable him—depending on his own common-sense—to do almost anything. It is quite usual in some polytechnics to give an elaborate course of many months, sometimes of a year; every problem ever worked by anybody must be done by every student. Every solution of nearly every problem has got tacked on to it the name of the professor who did it. It is the same with descriptive geometry and other branches of graphical mathematics. The average student properly taught during a few weeks is really a master of each of these subjects, whereas the student who has had an elaborate course retains no initiative and can attack no new problem. If time allowed I could state some amusing illustrations of this.

It is this want of common-sense which makes the usual school and polytechnic and university courses in applied science so elaborate. I have here an advertisement, covering the last large page of a daily paper, of all the courses in a certain foreign polytechnic. To prepare to enter, I may say that a boy must work very hard indeed till he is nineteen years of age. It is not at all unusual to find that at eighteen a boy has to stop working for a year because his health has broken down. I know that it is usual to find these boys ignorant of anything even slightly outside their school course.

A spiritless boy of nineteen, with rounded shoulders, tells you that he has done much science, and *all* algebra, and *all* trigonometry, and you feel proud that English boys stolidly refuse, in spite of all punishment, in spite of being called dunces, to become such products of academic machinery.

This polytechnic syllabus is well worth study. It begins with

the stupefied boy of nineteen, and it gives him four years of pure and applied mathematics of all kinds, and after a practical course he is a finished civil, or mechanical, or electrical, or some other kind of engineer or architect.

Now I am an advocate of technical education as it is given in a few places in England, and as I hope that it will be given in many places when boys come to be properly prepared from school; but I solemnly affirm that an English boy with the usual ignorance of mathematics and physical science when leaving school, pitchforked into a workshop where nobody thinks it his duty to give him instruction; ignorant of theory all his life, getting no scientific education whatsoever, cannot be a much worse engineer than the product of such a polytechnic as this.[1] Can these students ever get back their birthright that they have fooled away? Workshop experience, contact with workmen and nature, must give them some common-sense, but it is difficult to see how they can ever get to think for themselves, to act without explicit orders. Can they ever invent? Can they ever become free men? I have seen them trying to become men by sedulously neglecting, forgetting, and contemning all the learning of their polytechnic.

On a very much smaller scale one may see the same thing in English Naval, and Artillery, and Royal Engineer officers. Every one of them will tell you that in general intelligence he

[1] I have taught and examined every kind of boy and man; I have worked with men in the shops as one of themselves; I have been a manager of works and an employer of labour; I have had long experience of the ways of all kinds of manufacturers and engineers. In spite of enormous loss continually going on through the ignorance of English-speaking engineers (I know of most dreadful examples), the great majority of English-speaking employers of engineers believe the unscientific English engineer to be preferable to the finished polytechnic engineer. As an employer I myself used to feel very undecided. Now, if there is even an approach to equality of value, see what a serious reproach there is upon the new system. Imagine polytechnic authority insisting on the learning of every detail of an elaborate life-course of study when not the smallest part of it is known to English engineers of practically the same actual value. I know by actual trial that with a quarter of the German fag we can give to the average young Englishman such a knowledge of scientific principles that they become part of his mental machinery, and he can no more forget them, or how to apply them, than he can forget how to read or write. Surely I may be forgiven if I urge strongly the enormous importance of the reform which I advocate.

must have greatly benefited by his mathematical training, although he may be unable to do any mathematical work, having forgotten everything that he ever learnt. He has the sort of respect for science that a famous ruffian of Charles the Second's time had for religion, who stated that he always saluted a church by uncovering his head to it.

All advocates of orthodox methods (keeping the examination form in the background) seem willing to sacrifice every form of usefulness of mathematics to one form, the emotional or soul-preserving mind-training inherent in a perfect logical system; a huge complex deduced logically from simple fundamental truths. It would take me too long if I were to dilate upon the fact that it is time to cease talking of certain things as being fundamental truths, and that all logical deductions from them must be correct. Unless I gave my reasons at considerable length, I might earn a character for flippancy, for sneering at one of the most wonderful and soundest structures built up by human beings out of mental chaos. But I have the right to exclaim against a worship of the structure which prevents its being preserved, enlarged, embellished, and made use of.

As soon as we give up the idea of absolute correctness we see that a perfectly new departure may be made in the study of mathematics. The ancients devoted a lifetime to the study of arithmetic; it required days to extract a square root or to multiply two numbers together. Is there any great harm in skipping all that, in letting a schoolboy learn multiplication sums, and in starting his more abstract reasoning at a more advanced point? Where would be the harm in letting a boy assume the truth of many propositions of the first four books of Euclid, letting him accept their truth partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra? Letting him assume the sixth book to be axiomatic? Letting him, in fact, begin his severer studies where he is now in the habit of leaving off? We do much less orthodox things. Every here and there in one's mathematical studies one makes exceedingly large assumptions, because the methodical study would be ridiculous even in the eyes of the most pedantic of teachers. I can imagine a whole year devoted to the philosophical study of many things that a student now takes in his stride without trouble. The present

method of training the mind of a mathematical teacher causes it to strain at gnats and to swallow camels. Such gnats are most of the propositions of the sixth book of Euclid; propositions generally about incommensurables; the use of arithmetic in geometry; the parallelogram of forces, &c.; decimals. The camels I do not care to mention, because I am in favour of their being swallowed, and indeed I should like to see them greatly increased in number: they exist in the simplest arithmetic, and geometry, and algebra. Why not put aside ever so much more, so as to let a young boy get quickly to the solution of partial differential equations and other useful parts of mathematics that only a few men now ever reach? I have no right to dictate in these matters to the pure mathematicians. They may see more clearly than I do the necessity for a great mathematician going through the whole grind in the orthodox way; but, if so, I hardly see their position in regard to arithmetic and other things in the study of which they do allow skipping. I should have thought that the advantage of knowing how to use spherical harmonics or Bessel functions at the age of seventeen, so as to be able to start in mathematics at Cambridge just about the place where some of the best mathematical men now end their studies for ever, of starting at this high level with youthful enthusiasm, and individuality, and inventiveness, would more than compensate for the evils of skipping.

I might have put all this in the following briefer form. Great fields of thought are now open which were unknown to the Alexandrian philosophers. If we begin our study as the Alexandrian philosophers did, with their simplest ideas in arithmetic and geometry, we shall get stale before we know much more than they did. If we begin assuming more complex things to be true (although I do not like to assume that in truth any idea is more complex than another), as we have done in arithmetic, as we ought to do in other parts of mathematics, without becoming stale we may know of all the modern discoveries. We shall thus get the same intellectual training with more knowledge.

I have been speaking of the training of the mathematician, and I may be wrong; but, as to the educational training of the man who is to use his mathematics in the study of pure and applied

physical science, I have no doubt whatever of the importance of skipping greatly in all early mathematical work.

In these days all men ought to study Natural Science. Such study is practically impossible without a knowledge of higher mathematical methods than that of the mere housekeeper. It must be more than what is called "knowledge"; it must be mental dexterity, and it must be kept in constant practice if it is not to become rusty, and if men are to remain unafraid of mathematics. As examples of methods necessary even in the most elementary study of nature I may mention:—the use of logarithms in computation; knowledge of and power to manipulate algebraic formulæ; the use of squared paper; the methods of the calculus. Dexterity in all these is easily learnt by all young boys. In such practice their brain power develops quite rapidly, and they learn with pleasure. I feel sure that such dexterity cannot hinder, and can only further the mathematical study of the exceptionally clever student.

For an advanced study of natural phenomena we need the results of the best study of the greatest mathematicians.[1] To

[1] I believe that the useful methods o. mathematics are easily to be learnt by quite young persons, just as languages are easily learnt in youth. What a wondrous philosophy and history underlie the use of almost every word in every language—yet a child learns to use the word unconsciously. No doubt when such a word was first invented it was studied over and lectured upon, just as one might lecture now upon the idea of a rate, or the use of Cartesian co-ordinates, and we may depend upon it that children of the future will use the idea of the calculus, and use squared paper as readily as they now cipher. As I said at the Society of Arts, in 1880:—"When Egyptian and Chaldean philosophers spent years in difficult calculations, which would now be thought easy by young children, doubtless they had the same notions of the depth o. their knowledge that Sir Wm. Thomson might now have of his. How is it, then, that Thomson has gained his immense knowledge in the time taken by a Chaldean philosopher to acquire a simple knowledge of arithmetic? The reason is plain. Thomson, when a child, was taught in a few years more than all that was known three thousand years ago of the properties of numbers. When it is found essential to a boy's future that machinery should be given to his brain, it is given to him; he is taught to use it, and his bright memory makes the use of it a second nature to him; but it is not till after-life that he makes a close investigation of what there actually is in his brain which has enabled him to do so much. It is taken in because a child has much faith. In after years he will accept nothing without careful consideration. The machinery given to the brains of children is getting more and more complicated as time goes on; but there is really no reason why it should not be taken in as

me, mathematics is a powerful weapon with which to unlock the mysteries of Nature. If a man knows how to use the weapon, that is enough. Let him leave to others, the men who delight in that, the forging of the weapon, the complete study of it. If I can use the weapon, let my study be of another kind—I think, perhaps, of a higher kind—to study the secrets which even an unskilful use of the weapon will reveal to me. Fain would I know more about how the weapon was made and how to forge it for myself; but if I have no delight or skill in making weapons, and if I have enormous delight in using them, then will I use them if I can, and practise using them till I become skilful, for I know that the weapon-maker is not likely to be skilful in its use.

I have the belief that the study of physical science, and therefore the study of mathematics, by everybody, however poor or however rich, is of the utmost importance to our country, not merely for the knowledge it gives, but for producing the

early, and used as readily, as were the axioms of childish education in ancient Chaldea. A watch is a complicated piece of mechanism, which it has taken the thought of all the ages to elaborate, but the smallest boy can make it useful to himself. In a recent number of *Macmillan's Magazine* there is a paper by the late Professor Clifford, on 'Boundaries.' It directs attention to some of the simplest mathematical ideas. I felt, when reading it, that nobody could take a greater interest in it than a mathematician who had long used those ideas. The notion of a boundary had long been simple to him, and useful, like his watch to a boy; but one day he looks into its mechanism, and without it becoming less useful, he finds that it opens up for him a world of thought." It is interesting to notice from this old paper of 1880 how clearly we saw, even then, the necessity for the reform that I now advocate. I venture to give one more quotation :—" In pointing out that, as time goes on, we must begin from more and more comprehensive data—in fact, that pupils must commence their studies farther and farther from the real beginning of the subject—I point to a fact of which every teacher of physics gets good evidence even in the history of twenty years. Teachers of mathematics have not this evidence; but the teaching of natural science is obeying natural laws as yet, not being fettered by crystallised rules and vested interests. Instead of teaching pupils, . . . and giving them lectures on virtual velocities, and the like, as was common ten years ago, we begin with a great generalisation, the law of conservation of energy. And yet, after this generalisation was known to men of science, how long it was before teachers ventured to prune away excrescences from the text-books; how long it was before they ventured to say, 'It is not necessary to teach as we ourselves were taught; we can do better; we can give in a few words, and illustrate by a few experiments, a general law which it required years for us to understand.'"

scientific habit of thought, giving to every unit of the population a power to think for itself, and so producing the greatest happiness, and giving the greatest strength of all kinds to the nation.

I believe that men who teach demonstrative geometry and orthodox mathematics generally are not only destroying what power to think already exists, but are producing a dislike, a hatred, for all kinds of computation, and therefore for all scientific study of nature, and are doing incalculable harm.

What I say is especially important for the great and increasing number of men whose occupations are with applied science. I am particularly interested in engineering, and have a particular knowledge of the wants of engineers. It must be remembered that the engineer is becoming a very important person. What is going to affect the rule of nations over one another more and more is that for which Napoleon gibed at us—manufacture, the development of all natural resources which help in manufacture, and the distribution of things manufactured. To perform this well requires character and power to think in every one who has to do with it, whether managers or foremen or workmen. It is becoming more and more evident that the engineer requires a scientific technical training, and we are face to face with the ghastly fact that our engineers young and old are unfitted by their school education to undergo this necessary training. Furthermore, although he feels his needs, bitterly, deeply, he scorns the idea that mathematics will help him in any way, for he has already done at school what was called mathematics. Such mathematics as he was taught is indeed a very useless thing to him. He wants to be able to use such very simple mental tools as the ideas of the calculus; he needs them in one way or another in every kind of engineering; he needs to be so familiar with them that he can use them in every kind of new problem that comes before him. There is no part of engineering of which the theory does not need the use of these simple tools. And we tell him that he cannot learn to use these tools until he has worked for many years on the studies of the Alexandrian philosophers and their followers. What I know for certain is that the average man cannot learn the use of these tools except when young. We have some excellent technical colleges at our universities and elsewhere, but the students who enter them are prepared to enter

only the older kind of technical college which prepared clergymen and literary men and lawyers and legislators for their professions; they are in no way prepared to enter science colleges. And so, many of these science colleges are merely marking time, or doing what is much worse, giving to their students an education in mere formulæ which they do not really understand, the power to think being utterly lost, so that the students of these colleges become the laughing-stock of the average engineer.

It is curious how early mathematical training, by inducing a belief in what looks like mathematical reasoning, prevents men from testing the value of the engineering "laws" that they read or listen to. Rules that are approximately true in only certain cases are supposed to be absolutely true in all cases. In some of our colleges the engineering professor begins by trying to undo the evils of early mathematical training, cultivating the common-sense of his pupils, letting them see the real value of mathematics, and in such colleges very good work is being done. I may say, however, that it is really too late in the life of the average boy to begin such training after he enters college.

One sometimes finds a good mathematician taking to engineering problems. But he is usually "stale" and unwilling to go so thoroughly into these practical matters, and what he publishes is particularly harmful because it has such an honest appearance. When we do get, once in thirty years, a fairly good mathematician who has common-sense notions about the things that engineers deal with, or a fairly good engineer who has a common-sense command of mathematics, we have men who receive the greatest admiration from the engineering profession, and yet it seems to me that quite half of all the students leaving our technical colleges ought to be able to exercise these combined powers if mathematics were sensibly taught in school and college.

Perhaps the worst fault of our teaching is that the pupil is taught as if he were going to be a teacher himself. A man who teaches a subject is kept constantly in mind of a hundred rules. A man who does not teach but who is able to utilise his knowledge remembers only a very few rules—he knows these so well by constant use of them that he can apply them to every possible case that arises.

Take pure or applied mathematics, for example. In any ordinary treatise at the end of every chapter there are, say, twenty examples, all to be done by the labour-saving rule or rules taught in the chapter. But a clever man who is not a teacher can do every kind of problem—possibly in a clumsy way, but he can do it—by the use of perhaps one or two general principles which he never forgets. The average man not having to teach, who has gone through treatises and passed examinations, forgets these hundreds of rules; he remembers his hard study, gets disheartened at the notion of re-studying what he has forgotten, and indeed gets to loathe the idea of it. He was not taught to look at things from a really practical common-sense point of view; to practise the use of one general principle in all sorts of quite different-looking problems. I repeat—the average young engineer may be made to possess a power of using the methods of mathematics which will be as easy to him as reading or writing or using any hand tool, a power which never grows rusty because he exercises it every day of his life. His present intense hatred of mathematics and all theory of engineering requiring a knowledge of mathematics is very dreadful and is leading towards disaster. It seems greatest in men who have been to good schools; it is noticeable in many men who have attended science classes in colleges; it is very noticeable in electrical engineers whose whole profession is based on mathematical computation. Engineers some of whom are Fellows of the Royal Society are quite outspoken in their condemnation of all theory, all computation more complex than that of the housekeeper, and they themselves are unable to distinguish between work and power; between coulombs and ampères. I do not now speak merely of the older men.

For the boy of brilliant intellect, of whose education mathematics forms a small part, to whom Euclidean reasoning is a logical counsel of perfection, even the shortest study of it is altogether good. For such intellectual training as he requires I feel sure that my improved method of study would be ever so much better. I admit, however, that this sort of boy gets no great harm under the present system. It is the average boy who is to be pitied, because he is stupefied in being forced to study things that have no meaning to him whatsoever. And even the fairly clever

mathematical man who becomes a teacher in other subjects is led by his mathematical study to be much too apt in teaching anything to begin with the abstract philosophy of the new study. In beginning to teach a boy to play whist, or to swim, or to write, or to count, or to play billiards, he would philosophise so much and introduce difficulties so unnecessarily that the boy would find it practically impossible ever to learn. He never gets to know that the proper method of teaching any subject is through some kind of experimental work. He is the sort of man who makes children begin to learn a foreign language by its grammar.

I find that a quite common way of beginning the subject of practical geometry is to give lectures on the philosophy of representing a distance *to scale*. That the distance of three feet may be represented to scale by a distance of one inch really needs no philosophic introduction. I find that no boy meets with any difficulty in comprehending what one means by the scale of a map or any drawing; but it is easy to create in him much mental confusion if we proceed to point out what difficulties he *ought* to experience. And if the philosophy of such a simple scale as this is difficult, think what it must be when one represents on squared paper a quantity such as "the price of silk per pound," or "the height of the barometer in inches" by a distance of one inch. Think of how one could reduce not only the minds of all our students, but the mind of one's self, to imbecility by philosophising on this subject. I find that if you tell a boy to represent any quantity whatsoever to scale, showing him in ten seconds what you mean, he does it, he understands you, and it is only as a much older man that he begins to see how occult he ought to have found the subject.

Like almost every subject of human interest, this one is just as easy or as difficult as we choose to make it. A lifetime may be spent by a philosopher in discussing the truth of the simplest axiom. The simplest facts as to our existence may fill us with such wonder that our minds will remain overwhelmed with wonder all the time. A Scotch ploughman makes a working religion out of a system which appals a mental philosopher. Some boys of ten years of age study the methods of the differential calculus; other much cleverer boys working at mathematics to the

age of nineteen have a difficulty in comprehending the fundamental ideas of the calculus. I wasted much precious time of my life on [the subject of] the fifth book of Euclid, and most people approach the sixth book through years of worry over the earlier geometry, but indeed almost all the propositions of the sixth book may be taken as axiomatic. I know men who seem as if they wanted to revert to the Greek method of dealing with arithmetic. I know teachers who complicate practical geometry work by extracting square roots and performing other simple arithmetical operations, and set such problems in examination papers as: Find

$$2\sqrt{3}+\sqrt{2}.$$

They might be forgiven if they did not happen to be utterly wrong from the philosophic point of view which they fancy they occupy.

In most of what I have said I have been considering those boys who take readily to abstract reasoning. I wish to refer more particularly now to the average boy, who represents 99 per cent. of all boys, the boys who dislike abstract reasoning.

If a method of study is disliked by a pupil, a certain kind of knowledge may be given and the power to pass examinations; in this aspect there may be an advantage to the pupil; but there is no training of the mind except in the direction of dulness and stupidity.

A healthy English boy by resorting to excessive athletics and by sheer obstinacy resists the evil, and even when everybody has called him stupid so often that he is quite convinced of his own stupidity, it has no great effect on his conduct. And all the time his teacher, who when young took kindly to abstract reasoning, cannot see that the average boy has quite a different way from his own of looking at things, and is really not at all stupid. He is sharp enough outside the mathematical class-room; he shows no stupidity afterwards in business, as a legislator, as an engineer.

In the whole history of the world there was never a race with less liking for abstract reasoning than the Anglo-Saxon. Every other race has perfected abstract schemes of government. No

other race has such an illogical law as our Toleration Act; and yet what a good law it is. Common-sense and compromise are believed in, logical deductions from philosophical principles are looked upon with suspicion, not only by legislators, but by all our mcst learned professional men.

When English visitors like Colet were privileged to meet Lorenzo and his friends in the Rucellai gardens, they also enthused over fresh finds of Greek manuscripts and over the philosophy of Plato, but it was then remarked that what they really brought back to England was just such knowledge as could be made of practical use. Always England has been less abstract than other countries. She has possessed mental philosophers, but the average Englishman tries to make use of his knowledge.

In this connection it is interesting to note that the complacent admiration of English engineers for the absence of mathematical theory from their profession is somewhat like that of English lawyers for the absence of all logic and philosophical principles from their technical and subtle system. Scientific men and psychologists hate to look into what they call our disgraceful engineering manuals and digests of the law. If English engineers had the same power over their trade as the lawyers have over theirs, no mathematics or physical science would ever be necessary for them.

No words oi any English play are ever so enthusiastically applauded as Sir Peter Teazle's "Damn your principles, sir." No words have such an instantaneous effect in bringing the sympathy of an audience as "I am a practical man, gentlemen." If, then, our Tom Browns and Tom Tullivers really represent 99 per cent. of Anglo-Saxon boys, is it not a pity that educational systems should refuse to recognise the fact?

I should like to put my views as to the two kinds of English boy in the following two parallel statements:—

The average English boy takes unkindly to abstract reasoning, and if compelled to such study when unwilling is hurt mentally for life; loses his self-respect first, then his re-

Even for exceptional young boys demonstrative geometry is bad educationally because they reason about geometrical magnitudes before they know what these magnitudes really are;

spect for all philosophy; gets to hate mathematics.

they apply the same reasoning to more complex ideas of which they have the same ignorance; they become vain of their specious knowledge; they get to hate all applications of mathematics.

Philosophy was never intended as a study for children, and even exceptional children do not really learn the Alexandrian philosophy; they only get expert in solving puzzles. It needs age and also a knowledge from experience of the ideas to which the philosophy is applied.

Whether or not all men ought to attempt philosophical studies is a question I need not answer. The old Greek answer is that only a very few are capable. But I do say that, both for those who are dull and for those who are quick, it is important not to begin such study too soon.

It is not in my present brief to say whether the boy who takes kindly to abstract reasoning is really the mental superior of other boys. We are all agreed, I suppose, that if such a boy has also other mental qualities he may be intellectually great. It is, however, well to remember that the easier understanding of, say, Euclid I. 4 or 5 may imply not greater, but really less mental power, and that the early and seemingly simple proofs in Euclid are really difficult of comprehension to a mind of great natural power. A boy with good reasoning powers suspects a trick, thinks there must be some hidden meaning which he fails to grasp, and it may be that he could deal much more easily with much more involved reasoning than that of the early propositions of Euclid.[1]

When I am told that mathematics can only be learnt through a lifetime, and must remain unknown to the average boy to

[1] Nov. 13th, 1901.—I have stated elsewhere my own experience in trying to give mathematical ideas to some exceptionally clever boys. Mr. X. is my favourite modern writer. His dialogues are almost too clever. His intellect is exceptionally great. I am told to-day by my friend Y. that he used to make attempts to give X. an idea of an angle, of angular magnitude, and he utterly failed. I also should fail if I tried to make the ordinary mathematical teacher comprehend this psychological fact.

whom it might be useful, I think of the time when men were able to do all their daily work without reading, writing, or ciphering; these were then the learned studies of lifetimes. But as soon as these were needed in people's daily work they were taught quite readily to children without unnecessary philosophy. And now, a child learns to compute long before it philosophises on number. Even Max Müller learned to speak and write long before he was taught grammar or philosophised on philology. M. Jourdain pronounced French very well before he was taught anything of the science of language. Even a poor training-college pupil has done some thinking for herself, and has had emotions, before she is compelled to begin courses on (save the mark) psychology and ethics.

Higher mathematics has got to be a very useful thing; what I argue is that, as in the case of all other generally useful things, the complete study of its philosophy in the orthodox manner is not a necessary part of a school or college curriculum.

My engineering friends think that I have an exaggerated notion of the importance to all men of possessing a love for mathematics. But they have not had my experience; they have not seen, as I have seen, that this affection can be produced in the average boy if he can be subjected to a delightful training when quite young; they have not seen its usefulness all through a man's life as I have seen it. I know of only one other thing that seems to be of equal importance in a man's education all through his life, and this is that he shall have got fond of reading all manner of books when quite young. Take almost any child who hates books, place it in a household where everybody is fond of reading, and it also gets fond of reading in a very short time.

Let me know that a boy possesses these two kinds of affection at the age of ten, and I know that, later, he will possess two great powers: that of being able to use mathematics and that of being able to use books. I have no time to describe what I mean by these two powers; I mean much more than might appear to a man who may have received only the orthodox scientific and literary training.

Such a boy's education will be a constant delight to his teachers if they will only refrain from prosing and the setting

of tasks; if they merely make timely suggestions and answer his questions, and leave him to find out things for himself.

Gentlemen, I think the present method of so-called education all over the world to be utterly unscientific. I dare not venture to express my feelings as to the effect which might be produced on the whole world by a reform in the teaching of mathematics, because I wish to appear to you a non-visionary, practical person who needs your votes. There can be no harm in saying that, as there is no real study of natural science which is not quantitative, it must be through mathematics. A man in the twentieth century, whose eyes are not educated through the principles of natural science, can take no proper lessons from history or literature. His imagination is dwarfed. He is a bad citizen because he is at the mercy of quacks of all kinds.

I maintain that the safety of a country is founded on the good education, the complete mental and physical development, not merely of a few, but of all its inhabitants. I can imagine a real education going on from childhood to old age. I can imagine citizens worthy of further developing and utilising the wondrous scientific discoveries of the world, of utilising the results of the labours of historians and philosophers; citizens enjoying poetry and all kinds of literature. Without the individuality of thought and inventiveness produced by true education in all people, one or two great races of the world may become more powerful than the Assyrians, or Persians, or Greeks, or Romans; but, as in those older races, each citizen will become a manufactured unideaed article, the creature of a system which must fall to pieces some time; and when this takes place the ruin will be so terrible, and the chaos will last so long, that our own past dark ages will seem to be insignificant in comparison.

## A Course of Elementary Mathematics.

*A course of study recommended for training colleges and for boys and girls. Preferably taken as part of the science course; that is, mixed with it.*[1]

*Arithmetic.*—Decimals to be used from the beginning; the fallacy of retaining more figures than are justifiable in calculations involving numbers which represent observed or measured quantities. Contracted and approximate methods of multiplying and dividing numbers whereby all unnecessary figures may be omitted. Using rough checks in arithmetical work, especially with regard to the position of the decimal point.

The use of $5{\cdot}204 \times 10^5$ for 520400 and of $5{\cdot}204 \times 10^{-3}$ for ·005204. The meaning of a common logarithm: the use of logarithms in making calculations involving multiplication, division, involution, and evolution.[2] Calculation of numerical values from all sorts of formulæ however complex.

The principle underlying the construction and method of using a common slide rule; the use of a slide rule in making calculations. Conversion of common logarithms into Napierian logarithms. The calculation of square roots by the ordinary arithmetical method. Using algebraic formulæ in working questions on ratio and variation. Simplification of fractions. Calculation of percentages. Expressing shillings and pence as decimals of a pound; quarters and pounds as decimals of a hundredweight or ton, &c., so that all problems in Practice, Interest, Discount, &c., become mere common-sense applications of the simple rules.

*Algebra.*—To understand any formula so as to be able to use it if numerical values are given for the various quantities. Rules of Indices.

Being told in words how to deal arithmetically with a quantity,

[1] Students of this Syllabus will do well to consult *A Summary of Lectures on Practical Mathematics*, published by the Board of Education, 1889.

[2] I should like to say that, since the Science and Art Department has distributed its tables of four-figure logarithms and functions of angles over the country as cheaply as grocers' advertisements, there has been a wonderful development in knowledge and use of such tables.

to be able to state the matter algebraically. All this has already been stated under the head of Arithmetic. Problems leading to easy equations in one or two unknowns. Easy transformations and simplifications of formulæ and in easy cases finding any one of several quantities in a formula when the others are given. Practice in algebraic manipulation generally.

The determination of the numerical values of constants in equations of known form, when particular values of the variables are given. The meaning of the expression "A varies as B."

Factors of such expressions as $x^2-a^2$, $x^2+11x+30$, $x^2-5x-66$.

*Mensuration.*—Testing experimentally the rule for the length of the circumference of a circle, using strings round cylinders, or by rolling a disc or a sphere. Inventing methods of measuring the lengths of curves. Testing the rules for the areas of a triangle, rectangle, parallelogram, circle, ellipse, surface of cylinder, surface of cone, &c., using scales and squared paper. Propositions in Euclid relating to areas tested by squared paper; also by arithmetical work on actual line and angle measurements. The determination of the areas of an irregular plane figure (1) by using the planimeter; (2) by using Simpson's or other well-known rules for the case where a number of equidistant ordinates or widths are given; (3) by the use of squared paper when equidistant ordinates are not given; finding such ordinates; (4) weighing a piece of cardboard and comparing with the weight of a square; (5) counting squares on squared paper to verify rules. Rules for surfaces of spheres and rings. Rules for volumes of prisms, cylinders, cones, spheres, and rings, verified by actual experiment; for example, by filling vessels with water or by weighing objects of these shapes made of material of known density, or by allowing such objects to cause water to overflow from a vessel.

The determination of the volume of an irregular solid by each of the three methods for an irregular area, the process being first to obtain an irregular plane figure in which the varying ordinates or widths represent the varying cross sections of the solid. Determination of weights from volumes when densities are given.

Stating a mensuration rule as an algebraic formula. In such

a formula any one of the quantities may be the unknown one, the others being known. Numerical exercises in mensuration.

The experimental work in this subject ought to be taken up in connection with practice in weighing and measuring generally; finding specific gravities, illustrations of the principle of Archimedes, the displacements of floating bodies, and other elementary scientific work. A good teacher will not overdo this experimental work; he will preserve a proper balance between experimental work, didactic teaching, and numerical exercise work.[1]

*Use of Squared Paper.*—The use of squared paper by merchants and others to show at a glance the rise and fall of prices, of temperature, of the tide, &c. The use of squared paper should be illustrated by the working of many kinds of exercises, but it should be pointed out that there is a general idea underlying them all. The following may be mentioned :—

Plotting of statistics of any kind whatsoever, of general or special interest. What such curves teach. Rates of increase.

Interpolation, or the finding of probable intermediate values. Probable errors of observation. Forming complete price lists by manufacturers. The calculation of a table of logarithms. Finding an average value. Areas and volumes, as explained above. The method of fixing the position of a point in a plane; the $x$ and $y$ and also the $r$ and $\theta$, co-ordinates of a point. Plotting of functions, such as $y = ax^n$, $y = ae^{bx}$, where $a$, $b$, $n$ may have all sorts of values. The straight line. Meaning of its *slope*, slope of a curve at any point in it. Rates of increase, illustrated by the speed of a body. Easy exercises on rates of increase of $y$ with regard to $x$ in the case of $y = ax^n$, with illustrations from mechanics and physics.

Determination of maximum and minimum values. The solution of equations; very clear notions of what we mean by the

[1] I may also hope that a good teacher will educate the hands and eyes of his pupils, so that they may become expert in guessing the weights of objects, and small and large distances. Absurd questions ought not to be set, because they teach a boy not to exercise his common-sense. There is the well-known university case of a man who was calculating how many postage stamps would be required to cover the walls of a large room, and his answer was 1·203 . . . .; I am told that he had more than twenty decimal places. This kind of exhibition, not of faith in mathematics, but of mere stupidity, is very common in young engineers who have never made experiments.

roots of equations may be obtained by the use of squared paper. Determination of laws which exist between observed quantities, especially of linear laws. Corrections for errors of observation when the plotted quantities are the results of experiment.

In all the work on squared paper a student should be made to understand that an exercise is not completed until the scales and the names of the plotted quantities are clearly indicated on the paper. Also that those scales should be avoided which are obviously inconvenient. Finally, the scale should be chosen so that the plotted figure shall occupy the greater part of the sheet of paper; at any rate, the figure should not be crowded into one corner of the paper.

*Geometry.*—Dividing lines into parts in given proportions, and other experimental illustrations of the sixth book of Euclid. Measurement of angles in degrees and radians. The definitions of the sine, cosine, and tangent of an angle; determination of their values by graphical methods; setting out of angles by means of a protractor when they are given in degrees or radians, also when the value of the sine, cosine, or tangent is given. Use of tables of sines, cosines, and tangents. The solution of a right-angled triangle by calculation, and by drawing to scale. The construction of any triangle from given data; determination of the area of a triangle. The more important propositions of Euclid may be illustrated by actual drawing; if the proposition is about angles, these may be measured by means of a protractor; or if it refers to the equality of lines, areas, or ratios, lengths may be measured by a scale, and the necessary calculations made arithmetically. This combination of drawing and arithmetical calculation may be freely used to illustrate the truth of a proposition. A good teacher will occasionally introduce demonstrative proof as well as mere measurement.

The method of representing the position of a point in space by its distances from three co-ordinate planes. How the angles are measured between (1) a line and plane; (2) two planes. The angle between two lines has a meaning, whether they do or do not meet. What is meant by the projection of a line or a plane figure on a plane. Plan and elevation of a line which is inclined at given angles to the co-ordinate planes. The meaning of the terms "trace of a line," "trace of a plane."

The distinction between a scalar quantity and a vector quantity. Addition and subtraction of vectors. Experimental illustrations.

In setting out the above Syllabus, the items have been arranged under the various branches of the subject.

It will be obvious that it is not intended that these should be studied in the order in which they appear; the teacher will arrange a mixed course such as seems to him best for the class of students with whom he has to deal. A good teacher must understand that no examination made by any one other than himself can be framed which will properly test the result of his teaching. He must endeavour to give knowledge which becomes part of his pupil's mental machinery, so that the pupil is certain to apply it in all sorts of practical problems, and will no more allow it to become rusty than his power to read or write or walk.

## Advanced Course.

The instruction includes greater elaboration of the work specified in the elementary course, that is, much more practice in such computation from more complex formulæ. Demonstrative Geometry based upon Euclid.

The use of approximate formulæ such as

$$(1 + a)^n = 1 + na \text{ when } a \text{ is small compared with } 1.$$

Rules in Arithmetic (as of compound interest, &c.) and in Mensuration, stated as algebraic formulæ. Any one of the quantities in a formula may be the unknown one.

Practice in the simplification of algebraical expressions. Solution of equations and problems leading to equations. Resolution of a fraction into partial fractions.

TRIGONOMETRY.—Some knowledge of such limits as $\sin\theta \div \theta$. How to find the values of the sine, cosine, and tangent for angles greater than 90°; complementary and supplementary angles.

Fundamental relations such as $\sin^2\theta + \cos^2\theta = 1$.

Calculating the values of $\sin x$, $\cos x$, $e^x$ and $\log x$ using series.

The fundamental formulæ for the sine and cosine of the sum or difference of two angles, that is

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

and the others. Formulæ derived from the above, such as those for the sum and difference of two sines or cosines, and those which connect an angle and the double angle.

The sine rule, or $\frac{\sin A}{\sin B} = \frac{a}{b}$ in triangles. Also the rule $c^2 = a^2 - 2ab \cos C + b^2$.

The expression for the area of a triangle, having given two sides and the included angle, $\frac{1}{2} ab \sin C$.

The truth of such formulæ ought to be illustrated numerically and graphically by taking numerical values of the quantities.

Mensuration.—Guldinus' Theorems relating to areas and volumes of surfaces and solids of revolution. Exercises on the area of a segment and sector of a circle, the area of the surface of a sphere between any two parallel planes; approximate rules for length of a circular arc.

Finding centres of gravity using squared paper.

Use of the calculus to find areas and volumes.

Use of Squared Paper.—The plotting of functions including such as

$y = ax^n$; $y = ae^{bx}$; $y = a \sin (cx + d)$; $y = ae^{bx} \sin (cx + d)$.

Having given observed values of two varying quantities which are known to follow one or other of laws like $pv^n = c$, $y = a + bx^2$, $axy = bx + cy$, to find the probable values of the constants.

When two varying quantities are known to follow a given but somewhat complex law, to determine a simple law which between certain limits will give values approximating to the correct ones.

Solving equations by the use of squared paper.

Maximum and minimum problems.

Rates and Sums.—Rate of increase of one quantity relatively to that of another; approximate method of calculating a rate of increase, as, for example, in the case where simultaneous values of two varying quantities have been observed experimentally, or by finding the *slope* of the curve obtained by plotting such values.

The term "differential coefficient" as applied to a rate or increase; and the symbol for it, namely $\frac{dy}{dx}$, where $y$ and $x$ represent the two varying quantities.

Rules for finding the differential coefficient of $y$ with respect to $x$, that is, $\frac{dy}{dx}$, when $y$ and $x$ are related in the following ways :—

$$v = ax^n ; y = ae^{bx} ; y = \sin x ; y = \cos x ; y = a \sin (bx + c) ;$$
$$y = \mathrm{A} \log (x + a).$$

The study of these functions.

Proof and use of the rules for differentiating a product of two functions or the function of a function. Successive and partial differentiation. Integrating by parts and by substitution and other simple devices.

Calculation of maximum and minimum values.

Integration regarded as the inverse of differentiation, or as a process of summation ; the symbols

$\int y\, dx$ and $\int_a^b y\, dx$ ; rough methods of finding an approximation to $\int_a^b y\, dx$

when numerical values of $y$ and $x$ are known. Integration of $y$ when tabulated for equal increments in $x$.

The expressions for the following integrals :—

$$\int ax^n\, dx ; \int ae^{bx}\, dx ; \int \frac{\mathrm{A}}{x + a}\, dx ;$$
$$\int \mathrm{A} \sin (ax + b)\, dx ; \int \mathrm{A} \cos (ax + b)\, dx.$$

The solution of simple differential equations.

In following the syllabus, to make students take an interest in the work, constant use ought to be made of illustrations from Mensuration, Mechanics, and Physics.

Geometry.—How the position of a point in space is defined by its rectangular co-ordinates $x, y, z$, or by its polar co-ordinates $r, \theta, \phi$ ; the relations between $x, y, z$ and $r, \theta, \phi$.

Determination of the three angles $\alpha, \beta, \gamma$ which a given line makes with the three co-ordinate axes ; the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Determination of the angles between a given line and each of the co-ordinate planes.

When a plane is given by its traces, to determine its inclination to each of the three co-ordinate axes and planes.

The above may be treated analytically or graphically.

Representation by its projections on the three co-ordinate planes of a line whose position and real length are given.

Determination of the angle between two given lines; the angle between two planes whose traces are given. Represent by its projections the line of intersection of two planes whose traces are given.

VECTORS.—The scalar product and vector product of two given vectors, with illustrations. Easy Vector Algebra.

## DISCUSSION.

PROFESSOR HUDSON: I think there is no subject the teaching of which is more important, believing as I do that mathematics constitute one of the most important of the early subjects of our school curriculum, and feeling that the bad teaching of mathematics reacts powerfully upon the learning of all other subjects. If mathematics were better taught the study of all other subjects would improve. That is one of the main reasons for endeavouring to effect a reform in the teaching of mathematics. With the reasons for teaching mathematics enumerated by Professor Perry I agree, with the exception of No. 4, and I am not quite sure I understand No. 8. All the others, however, seem to be good and worthy objects. But to attain them mathematics must be taught as a process of reasoning. Of the preliminary subjects of education it is the one that most chiefly cultivates the reason, and I conceive that mischief is done by teaching it in a manner possibly appropriate to the teaching of other subjects (though I doubt if it is so); namely, by means of the cultivation of memory only. I do not know to what extent Professor Perry will agree or disagree with me in that. He seemed to point to the unwisdom of teaching mere processes which are not understood. Geometry is perhaps the very worst case. It is taught sometimes merely as an exercise of memory. I lay it down as a sort of axiom that

the understanding of the pupil ought to be exercised in all his studies [in everything that he learns], and I am a little afraid whether what is described as skipping may not be interpreted to mean a weakening of the application of that doctrine. If the skipping means that the pupil is to pass on to things which he does not understand, which require something previous to be learned, then, I think, it is not to be commended; but if it means merely leaving out comparatively unimportant parts of the subject, but keeping to the main road, then, I think, it may be approved, and may be recommended to some kinds of pupils. Another point, I think, should be borne in mind more than it is. The teaching of elementary mathematics should be conducted so that the way should be prepared for the building upon them of the higher mathematics. The teacher should always bear in mind and look forward to what is to come after. The pupil should not be taught what may be sufficient for the time, but will lead to difficulties in the future. I am sorry that I have not had time to read Professor Perry's slips. I think the fault in teaching arithmetic is that of not attending to general principles and teaching instead of particular rules. Arithmetic is divided into an enormous number of chapters, and each rule is taught separately without showing the few general principles that are needed for them all. That kind of skipping I entirely disagree with, and I hope Professor Perry and I are not in want of accord there. I think that mental arithmetic should always precede written arithmetic. The processes are different, and it is a very great mistake for a person to try mental arithmetic by forming a picture of what he would do in writing. The difficulty Professor Perry has pointed out of the want of understanding of decimals is, I think, to be traced distinctly to the want of teaching of the principle of local value which is at the basis of our written arithmetic. Algebra should be based on arithmetic, and the general principles of algebra, expressing the general principles of arithmetic, should be taught. I think that the difficulties of teaching algebra are very much like those of arithmetic, and arise from giving people rules and processes for each particular section of the subject, and not showing how they are based upon general principles. I am inclined to attack the teaching of mathematics on the grounds that it does not dwell sufficiently on a few general axiomatic

principles. I quite agree with Professor Perry in the importance of early practical applications. So soon as the pupil has any knowledge at all, he should be encouraged to apply it to practical purposes. One paragraph in the lecture I thoroughly agree with, and that is that frequently the pupil gets a distaste for the study because he is reasoning about things he knows nothing about.

In arithmetic, the child should be taught to count objects instead of rehearsing the names of numbers. He should learn the first few numbers and their relations by handling counters. He should thus learn the primary addition first, and slowly and gradually make his own multiplication table, instead of learning it by heart, while he still needs to count on his fingers to add 6 to 7. He should learn much of the fundamental truths of arithmetic orally, before he commences any written arithmetic.

When writing begins, he should at first express the reasoning with great fulness, gradually being allowed to omit more and more of it. The most condensed and compact arrangement cannot be taught in the first instance. Methods should gradually improve as knowledge and facility are acquired.

Great as may be the difficulty and disadvantage of giving at a later age the teaching that is appropriate to an earlier, there is no royal road : the boy who has not had this Kindergarten play in the nursery, should have it at school. He can get on quicker than a baby. Any slowness at the beginning will be amply compensated for by rapidity afterwards.

Merely to teach a boy a fresh set of rules or an additional set, most useful though they may be in the workshop or office, while he does not understand the primary principles of the subject, will not effect an improvement in his education.

The first lessons in geometry may be given in play with bricks and other models. When the boy is able to wield a pair of compasses without danger, he should, as a sort of play, draw various figures : this will excite his curiosity to know why certain results, such as the bisection of a straight line, are attained, and prepare him for a course of geometrical reasoning.

A reform in the teaching of geometry is consistent with the retention of Euclid's order. No change of text-book will be efficacious unless the system of learning by heart is entirely given up.

Professor Forsyth: It is extremely difficult to attempt to do anything by way of covering even an appreciable portion of the range of this vast paper, mere extracts from which took Professor Perry thirty-five minutes to read. There are, however, one or two general remarks I would like to make. First of all, I cannot indulge in any of the strong rhetoric in which he has so freely indulged. He said he was not using extravagant language; but it seems to me that many of his sentences showed an extravagant want of appreciation of facts. He says that, as a method of teaching the higher mathematics, one of the ways of proceeding is to take the student as quickly as possible to the limits of all existing knowledge. I find in my chair at Cambridge that my standing difficulty, when I get a thoroughly well-trained student, is to take him to the boundary of knowledge in a small part of some section of my subjects. How it is going to be possible, as Professor Perry seems to desire, to take people to the boundary of all existing knowledge passes my comprehension. It seems purely impossible. Again, Professor Perry makes usefulness the main, if not the sole, test of the position of mathematics and of their development. I must point out what would be a platitude if we were not in discussion, that scientific subjects do not progress necessarily on the lines of direct usefulness. Very many of the applications of the theories of pure mathematics have come many years, sometimes centuries, after the actual discoveries themselves. The weapons were at hand, but the men were not ready to use them. Take the case of medicine, which surely is a practical subject. It owes immense debts to the study of sciences like physiology and bacteriology: yet these have been developed, and continue to be developed, along their own lines without being guided in the direction of immediate application at every turn. Yet, independent as has been their development, it is notorious that, perhaps all the more because of their freedom in growth, they have provided new knowledge that is of the utmost importance in the conduct of living processes. Take one last example, the X rays. If any one had been set down, as a practical problem, to take a photograph through solid things, I think the common answer would have been that he was being told to solve an insoluble problem. Yet its solution came from the physicists, indirectly as it were, in the course of researches

made to obtain knowledge for its own sake. The knowledge so obtained has subsequently led to wonderful results in its application. Influenced by these examples, and by others more directly mathematical upon which I shall not enter, I must decline to accept utility as the main or the sole discriminating test, either in the study or in the teaching of mathematics.

Something has been said about the use of mathematics in physical science, the mathematics being regarded as a weapon forged by others, and the study of the weapon being completely set aside. I can only say that there is danger of obtaining untrustworthy results in physical science, if only the results of mathematics are used; for the person so using the weapon can remain unacquainted with the conditions under which it can rightly be applied. I have some experience, by correspondence, of this practice on the part of workers, and have found, not once or twice alone, that an inquirer has taken some result in mathematics, without understanding it and without knowing its proper limitations, and has applied it to obtain physical results. The results often are correct, sometimes are incorrect; the consequence of the latter class of cases is to throw doubt upon all the applications by such a worker until a result has been otherwise tested. Moreover, such a practice in the use of mathematics leads the worker to a mere repetition in the use of familiar weapons: he is unable to adapt them with any confidence when some new set of conditions arises with a demand for a new method: for want of adequate instruction in the forging of the weapon, he may find himself, sooner or later in the progress of his subject, without any weapon worth having.

I might spend time in traversing many of Professor Perry's statements; but, in the limited time at the disposal of speakers, it would perhaps be more useful if I were to give some indication of my own view as to the way in which the beginnings of mathematics should be taught. My reason for dealing with the beginnings is that I thoroughly agree with Professor Hudson in his expressed opinion that if advanced mathematics are to be taught, if anything beyond mere elements are to be taught, the beginnings must be taught in such a way that they are fairly grasped, and that they are not uninteresting. As regards the beginnings, I suppose we would be willing to accept the position that every schoolboy has

to do some arithmetic; and according as the boy has a facility for doing arithmetic, he will be prepared to go on to algebra and geometry. As regards algebra, many of the text-books used to be certainly very repellent, but I am not prepared, because the books were repellent, to say that the subject should not be taught, and that because my view of a text-book has changed from what it used to be, all teachers also ought to hold that changed view. As regards the method of teaching algebra, I would make it, in the earlier stages, as much a generalised arithmetic as possible. Results obtained by algebra would be verified by arithmetical instances; and the use of a formula would be indicated as including any number of instances. Elaborate (and to my mind wearisome) processes, useful for solving artificial combinations of difficulties, would be at least deferred; with a comparative beginner, progress towards new ideas or new stages of old ideas can, I think, best be made by the simplest instances. And it is on this account that I would build algebra entirely on arithmetical foundations so far as concerns the teaching of beginners.

Next we come to the question, the much more difficult one, of geometry. A great deal is said as though all those boys who are taught arithmetic should be taught geometry. I think it requires to some extent a special faculty to appreciate geometry; but that is a passing remark. The way in which I would teach it is by first securing for it some foundations in the process of the most elementary education. Most of us would agree with the customary declaration, that what we want to have our boys taught is reading, writing, and arithmetic. I would put in a very earnest plea that drawing should be added, in all cases. Every child is willing to draw in some form or other; and by taking proper opportunity an indication will be given of an appreciation of geometry. Before I began to teach a boy the subject of geometry, I would take care that the boy has shown some sort of facility in using ruler and compasses. He should be made to go through a certain amount of common practical geometry. I want him to be able to use his instruments and to do something that is accurate in the measurement of distance. It is only after a boy has undergone a certain amount of training, and shown a certain amount of facility in geometrical drawing, that in my opinion he ought to be turned on to the study of geometry; and the way in which he

should be turned on to the study of geometry would be to utilise the practical knowledge that he is already familiar with and continues to acquire. By guiding him among things with which he is already familiar, he may be enabled to pass from the practical working of his instruments to reasoning about the operations he has carried through. But Professor Perry may say that my suggested method is not taking him to Euclid. It may be so; at any rate, it is a desirable preliminary to geometrical reasoning, for it gives the materials about which the reasoning is to be made. Without being an ardent defender of the Elements of Euclid as a text-book, still I would make it the basis of the teaching of a pupil, at any rate at the present time. So far as I can make out, one of the reasons why the book is adopted so largely in our country is that there is no other book which teachers in general would use, and which finds general acceptance to the same extent as Euclid. Not that I would adhere to the original proofs, to the exclusion of all else: that is not done in any of the books that deal with the Elements. But the sequence is a recognised sequence, and, within that restriction, there is reasonable opening for any teacher's individuality. Of course I know that there are ardent defenders of Euclid who regard his work as being so completely logical that it should not be altered: I do not share their view, knowing the large number of assumptions implicitly made and never stated.

Speaking generally, I would help the boy to develop along lines of natural facility, recognising that he will need guidance. But if, after all help given, it turns out that algebra and geometry remain a mere miserable nothing to him, then I say that the time for training his intellect by these subjects is over, and that he had better be turned on to subjects more congenial to his possibilities.

Before ending these remarks, I should like to refer to a statement which has been made, to the effect that mathematics are very often taught as if the students were all to be teachers. I could wish it were so, because then we should have more intelligent teaching. But it is not the case: neither mathematics nor any other subject can be instilled into pupils as though they were to be teachers. Indeed, if one had to make an honest confession of opinion, it would be to say that the one thing in which we are

notoriously deficient in England is the proper training of teachers. A proper system of training teachers should be established, so that a teacher, when he began the exercise of his profession, would not have to devote his time to practising upon his earliest pupils the method of teaching that might agree with his particular temperament.

One word more. I have chosen in my remarks to adopt the invitation of Professor Perry to express some severe criticism, and to indicate hostility to those views with which I do not agree. But I hope that this audience will not suppose (and I know he will not suppose) that, because my remarks have been mainly of a hostile nature, I am out of sympathy with many of his aims and all his methods. Where divergence exists, it ought to be expressed in such a discussion as this is; and it is vain to spend our time in formal compliments to one another, which can effectively be exchanged in private.

MAJOR MACMAHON (President of Section A): I think Professor Perry regards all students as boys who are to become engineers, and looks upon all mathematical teaching as if the aim and object of everybody were to be an engineer. We are not all to be engineers, fortunately, and I think Professor Perry's taking that view leads him to a certain extravagance of language. We are asked to give up Euclid. Before we do that, we must have something else put before us. The Association for the Improvement of Geometrical Teaching has not advanced a proper substitute. I have tried to find out from those who have been prominent in that Society what it really proposes as a substitute for Euclid, but without success. I look upon Euclid as an instrument for the cultivation of the mind. I think it should be taught in conjunction with geometrical drawing. Professor Perry has talked about his exercises when he was in the Fifth Book of Euclid. I was more fortunate than the Professor, because I was never taught the Fifth Book. It was omitted from the curricula I went through, and I am astounded to find that Professor Perry was educated where the Fifth Book was seriously taught. In the teaching of algebra, I think—and I am here in perfect agreement with Professor Perry—that at the outset all algebra students should be taught the foundations of numerical magnitudes

on *squared* paper, and in that way algebra, and also trigonometry, would form a suitable introduction to geometry, which, I think, should be taught before geometrical conic sections. I think solid geometry, trigonometry, and spherical astronomy should be taught together.

E. M. Langley: While differing from Professor Perry in many points, I quite agree with him as to the necessity for considerable modifications in our methods of mathematical teaching, at any rate in the earlier stages, and especially in geometry. No words—not even Professor Perry's words—can condemn too strongly the practice of beginning the study of geometry by "learning Euclid," as it is called. When a batch of new boys is handed over to us to "begin Euclid," we are confronted by difficulties from which we might well shrink if we did not know that they have been and are habitually overcome (only overcome, however, as a rule, at an unspeakable cost to both pupil and teacher). We have to train ordinary boys who have previously seldom reasoned formally and continuously about anything—(1) to reason formally and continuously, (2) about things of whose names and properties they are ignorant.

With regard to the first of these difficulties—the want of development of the reasoning faculties—the mathematical teacher, as such, can produce little direct improvement. He must be content to take his pupils as he can get them, with their reasoning powers undeveloped, or possibly repressed, by the training they have had in other subjects. But the second difficulty—their want of familiarity with the objects about which they are to reason—comes more within his province. He may, while they are learning Euclid, do much by familiar and well-chosen illustrations, by the use of models, and by work with ruler and compasses, to encourage the growth of sound geometrical ideas in his pupils' minds. He may, however, be able to do much more than this: he may perhaps be able to arrange that before beginning their formal course of deductive geometry they shall go through a course of experimental geometry, and thus gain a knowledge, sound and thorough as far as it goes, of the subjects about which they are to reason formally later on.

[As to the need of such a course I do not think there can be

any doubt. Even if a child has been fortunate enough to have been taught in a good Kindergarten up to the age of seven or eight, and to have been taken by carefully trained teachers through a well-considered course, there still exists a great gap between this and his introduction to Euclid at the age of eleven. In these three or four years he is sure to lose much of what he had previously gained. What is wanted is a school course to bridge over or fill up this gap.]

But though there may be a general agreement as to the need of introducing a systematic course of experimental geometry, there will be a still more general agreement as to the difficulty of fitting it into the already crowded school curriculum. I can only see one practical way out of the difficulty; that is, to make it a part of the science teaching.[1] More hours will, undoubtedly, have to be given up to science, and I think some of these must be taken for the experimental course in mathematics. This course, however, need not be merely experimental; short trains of deduction might be introduced from time to time, the pupils being led to see how from the truth of one proposition already established by experiment another may be deduced by a simple exercise in logic, the new one in its turn being verified experimentally. After a course like this, pupils would come to their course of formal deductive geometry well stored with sound geometrical ideas, and capable of reasoning upon them. But besides a first experimental course which shall lead up to and prepare for the deductive one, there is need for a second one which shall accompany it, and give a similar experimental knowledge of subjects higher up in the school course and beyond it.

This might come partly in the mathematical and partly in the

[1] In the long list of mediæval philosophers whose methods Professor Perry decries, it will be noticed that the name of Roger Bacon does not occur, and it is interesting to observe the close agreement between some of the pointed phrases in *England's Neglect of Science* and what Bacon has to say on the same subject:—"Nam una est scientia quâ ignoratâ nulla alia sciri potest: et quâ scitâ possunt aliæ de facili edoceri . . . . et hæc est mathematica. Unde totius studii destructio est negligentia mathematicæ. Quoniam qui ignorat quantitates continuas et discretas et earum applicationes ad cæteras res et scientias ignorabit omnia. [*Et, quod pejus est, omnis homo ignorans hoc suam ignorantiam non potest percipere.*]" Does the Professor write to the *Electrician* with the *Opus Majus* before him, and translate into vigorous English Bacon's vigorous if not classical Latin?

scientific parts of the school time-table. Much of it might, in fact, come in odd quarters of an hour, and incidentally in connection with ordinary lessons without set periods.

For example, while the preparatory course should have so familiarised the pupils with the properties of lines, angles, triangles, and parallelograms, as to make progress through Book I. easy, simultaneously with the progress through Book I. I would have a course anticipatory of Book II., and so onwards.

[I said just now that the time required for this course might come partly from the mathematical and partly from the science hours. One other hour might be made useful for this purpose, and at the same time more effective for its own: that, namely, put down for geometrical drawing. This subject is likely to consist too often in the application of an ill-digested set of rules without any apparent underlying principles, and to be taught with too little reference to the theorems already obtained deductively. But much more might be made of it: Loci with their application to the solution of problems and Plane Perspective might be gradually introduced, care being taken that ideas should be induced to form themselves in the pupil's mind without any consciousness on his part that he was learning to use powerful scientific arms with formidable technical names.]

But while I look for considerable improvement to come from modifications of the school curriculum, I look for much more to come ultimately from the growth of sound ideas among parents, teachers, examiners, inspectors, and governors, [and from the establishment of system in our secondary education]. When Professor Perry did me the honour to suggest that I should take part in this debate, I took two steps that I thought might be helpful to me: (i) I made a thorough investigation of the work going on in our Kindergarten and in the Students' Training College attached to it; (ii) I procured Dr. Young's *Teaching oj Mathematics in the Higher Schools of Prussia.* As to (i), I cannot help thinking that much good must ensue from the excellent work going on in the classes for little ones, and in that for the ladies who are being trained to teach them, [work going on equally well, I have no doubt, in many other educational centres]. I should like to say more, but must pass on to Dr. Young's book. As this will probably have been in the hands of

most of you, I need do little more than refer to the five years' careful training insisted on as a qualification for a teacher, three in preparation of subject-matter, two for acquiring the art of teaching; but an incidental remark from a German colleague illustrates the difference between Prussian ways and our own. He had apparently been studying the progress of our various crews on the river, and had been struck with the fact that though the masters in charge of the boats seemed to say and do very little, yet the boats went continually faster and faster, and when I mentioned Dr. Young's book to him, he made the unexpected but suggestive reply: "Mathematics in Prussia! Ah, sir, they teach mathematics in Prussia as you teach your boys rowing in England: they are trained by men who have been trained by men who have themselves been trained for generations back." This seems to me to put the matter in a nutshell. But we should do little good by copying this part of the Prussian system if we do not go further. After all, an important part of a teacher's training comes, or should come, after his special years of preparation and study are over, and he should himself be continually growing. To quote Dr. Young: "If he is embarrassed by financial cares, and harassed by struggles to improve his financial position, his growth is retarded, and the quality of his work inevitably deteriorated. The teacher is spurred on to higher achievements by devotion to his calling and by the inspiration of seeking, finding, and imparting truth, and not by the competition of the mart, or by the goad of necessity. . . . The educational system of Prussia, while insisting on higher standards and severe tests at the outset, assures a tranquil career to those who give evidence of their fitness." They are not liable to be turned off at forty to attempt to get a living by poultry or fruit, or by editing a socialist newspaper. "The German teacher works with a sense of security—security in a modest competency while at work; security in the case of a rainy day; security that after a quiet life of fruitful toil he will reach a well cared for and honoured old age."[1]

[1] NOTE.—Two other points were raised on which I should like to add a few words: (i) the work of the Association for the Improvement of Geometrical Teaching; (ii) the nature of the Army examination papers.

As to (i), I was secretary to the A.I.G.T. for some years, and as such took an active part in a prolonged correspondence with University authorities and the Civil Service Commissioners as to their insistence on Euclid pure and

PROFESSOR EVERETT: The teaching of geometry has been too pedantic. The minds of boys and girls are not ripe for dealing with abstractions. The way in which Euclid begins (especially if the whole body of Definitions is taken first) gives the learner the impression of a castle in the clouds.

A moderate amount of practical geometry should come first, including methods of bisecting lines and angles, drawing lines at right angles, making a triangle with sides of prescribed lengths, and inscribing a regular hexagon in a given circle. This will give the learner definite conceptions, and help him to feel that he is on solid ground.

Side by side with Euclid or a substitute for Euclid, verification by actual measurement of carefully drawn figures should be encouraged. It is useful as a test of the accuracy with which measurements can be made by the methods employed, and also useful as a check against mistakes—which are liable to be made in abstract reasoning as well as in other matters. One of the most important habits in scientific investigation of all kinds is the habit of testing the correctness of one's conclusions by independent methods; and this habit should be inculcated by assiduous practice, as an important element in personal character—an element inseparably associated with the honest pursuit of truth.

simple. We gained what in those days was a considerable concession—the permission to use any proof that did not interfere with Euclid's sequence, but his sequence and his axioms were still retained. The Council of the A.I.G.T. asked for complete freedom as to text-books, and not for any special favour for its own. [For though Major MacMahon seems unaware of it, the Association did publish a text-book (Swan, Sonnenschein, & Co.), which has been used with marked success in the colonies, and accepted officially by some examining bodies here.] I was, and am, of opinion that we should have had more success if we had confined our requests to our own text-book. I fancy we might then have succeeded—at any rate at Cambridge, and possibly with the Civil Service Commissioners; as it was, I feel sure that the Council's liberality of sentiment was prejudicial to our cause. I think the first practical step would be to get the A.I.G.T. text-book accepted. When examining bodies find that no particular harm follows, further concessions might easily be asked for, and probably obtained.

(ii) One speaker specially attacked the Army examinations. Whatever faults they have had in the past, I know of no papers which have gone farther towards encouraging—I might almost say *enforcing*—some of Professor Perry's proposed retorms. I need only refer to the mathematical papers set last July and at the two previous examinations.

The learner should be taken on, as quickly as is consistent with intelligent progress, to the higher branches of mathematics. You understand the lie of a country better when you have surveyed it from a mountain-top; and you can work at the elementary parts of mathematics with more intelligent interest when you have found what they are leading you to. A certain amount of practice in elementary processes is necessary to prevent the student from stumbling at every step; but it is not necessary to exhaust one region before proceeding to the next. Too much time and labour are often devoted to subjects not necessary as means of further progress—for instance, to geometrical conics, and to very elaborate exercises in trigonometry. Students in the University of Cambridge, and in public schools preparing for the University, have for the past century been compelled to spend much time in laboriously learning how to do things without using the proper means. Students should as early as possible be initiated into the most effective ways of attacking problems. The elementary conceptions of the Infinitesimal Calculus, and its simpler processes, should be introduced at an early stage in mathematical teaching. The most essential points are—

1. A differential coefficient as a rate of increase, as a velocity, and as the tangent of a slope.

2. A definite integral as the area bounded by two ordinates.

3. The proof of

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n$ a positive integer.

4. The proof of

$$\int_a^b x^n = \frac{b^{n+1} - a^{n+1}}{n + 1}$$

for $n$ a positive integer; the proof being derived from considerations relating to the area between ordinates $a$, $b$, and to tangent of slope.

Another subject that is too long postponed is solid geometry. It is postponed so long that most boys do not get it at all. Considering that we live and move in space of three dimensions, it is unreasonable and unpractical to confine all accurate thinking and teaching for three or four years to two-dimensional space. The

result is to produce an instinctive shrinking from all three-dimensional thinking, as if it involved some terrible mystery. It should be taught in a realistic manner, with the aid of models; and the properties of direction-cosines should be introduced before the learner has got very far in trigonometry.

I agree with Professor Perry in recommending the use of squared paper for a variety of purposes.

The teaching of arithmetic is injudiciously directed by most of the examination programmes at present in use.

They err by excess in attaching too much importance to the arithmetical extraction of cube roots, and to the reduction of complicated circulating decimals to vulgar fractions—two processes which a practical calculator never employs—mere fancy subjects. On the other hand, they omit the practical use of logarithms for multiplication, division, raising of powers, and extraction of roots, though this is an everyday business with the practical calculator.

It is not necessary to teach any more of the theory of logarithms than is involved in the law of indices. The methods by which the tables have been computed is a matter of elaborate technical detail that does not concern the user; but expertness in using the tables to aid calculation should be insisted on. The expertness should include facility in taking out a four-figure logarithm and the number corresponding to a four-figure logarithm, from a seven-figure table.

In working examples, the general rule should be to carry logarithms to four figures only, as this gives what is for most purposes a high degree of accuracy.

Such practice will give readiness in the use of tables generally —a matter of great practical utility.

Another arithmetical subject that ought to be taught is the influence of small changes in the data. For instance, $x$ and $y$ being small, it is important to know that very approximately

$$(1 + x)(1 + y) = 1 + x + y,$$

$$\frac{1}{1 + x} = 1 - x,$$

$$\sqrt{1 + x} = 1 + \tfrac{1}{2}x,$$

$$(1 + x)^2 = 1 + 2x.$$

Another branch of approximate calculation that should be taught is contracted multiplication of decimals; and here, too much time should not be spent in ensuring accuracy in the last figure. Give and take should be permitted, to save time. The position of the decimal point should be determined by a very rough approximation, in which two numbers each having only one significant figure are multiplied together.

This illustrates the general use of rough calculation in round numbers to check mistakes in a more elaborate calculation; a subject to which much importance should be attached.

The whole subject of approximation seems to be overlooked in the teaching of arithmetic. It is necessary that boys should learn to deal with fractions in such a way as to be able to deduce the exact results to which the data lead; but it is equally necessary to acquire the knack of making reasonable approximations. In physical and engineering calculations the data are always approximate, and it is absurd to carry results to a much higher degree of exactness.

Clear arrangement of the steps in arithmetical calculations should be carefully taught, but not in a pedantic manner. The object should be to make the work easy of revision, and the memorandum style should be adopted in preference to full formal statements. Quantities to be added together should be arranged vertically with the sum underneath, not horizontally with *plus* signs between.

In logarithmic work, the formula written down as a guide should indicate multiplication, &c.; and underneath, without any logarithmic formula, the logarithms should be written one under the other, and added, &c.

Subordinate steps, which cannot conveniently be shown in full in the principal work, should be in some blank space not far away.

In *liberal* education (as distinguished from *technical*) those parts of the subject should be selected which are most fitted to give intellectual pleasure.

Æsthetic rather than utilitarian considerations should prevail. Information should be presented in scholarly shape —sharply defined, and clearly arranged, with due attention to neatness of expression both in words and notation; and the

pupil should be trained to cultivate similar qualities in his own work.

There is a want of the amenities of literature in many of the standard mathematical books. The reader feels himself snubbed; he is not taken into the confidence of the author, but treated as a working drudge rather than as a reasonable being. Strange-looking definitions should not be sprung upon a learner without the explanation which courtesy requires.

In conclusion, I wish to say that I have carefully scrutinised the questions set at the first South Kensington examination in practical mathematics, and worked out more than half of them—a larger proportion than candidates were allowed to take—with the result that I highly approve of the selection of subjects and the general style of the questions.

PROFESSOR RÜCKER (President of the British Association): There is very general agreement among all the speakers that in the case at all events of the younger children—and I think the same principle ought to apply to those of maturer years—it is well to approach the subject in the first instance as far as possible from the concrete side. That is the way in which a subject naturally presents itself to the mind of a child. When a student got to the advanced stage, it might be necessary to consider every step he takes. It is better that a student should be brought up rather rapidly to a point, and afterwards go into the various difficulties. Then he may be taught the various qualifications in order to protect it against all attack after the first general statement has been put before him.

One other remark. Professor Perry referred in his address to the teachers as the chief sinners in regard to examinations. I am not going to attack examinations; all the same, I do want to say that while they have their weak points they are a necessary part of our educational apparatus. I do not believe it is the teachers as a body who are to blame. It is a case of "my people will have it so"; it is because the English people have learned that examinations provide the only way for testing, not only the student, but the teacher. This condition has arisen not from the inside educational system, but by pressure brought to bear upon it from the outside.

PROFESSOR SILVANUS P. THOMPSON: As an examiner in physics in the University of London, I have experience of the deplorable stuff passed off as mathematics in the lower examinations of the University. It is all very well to blame the teacher and the teaching, but the fault is not there: the real fault lies with those who dictate to the teachers what and how they are to teach. The headmistress of a girls' school was so impressed with Sonnenschein's methods that they were used from the top to the bottom of the school for several years. But there came a day when an examiner, sent down by the Board of the High Schools, visited that school to examine the girls, and throughout the whole school wherever the Sonnenschein method had been used, either in compound multiplication or reduction, not only were the answers marked wrong, but marks were taken off. The result was that Sonnenschein's methods had to be abolished in that school because of the examiner. In two of the oldest universities in this country you are absolutely forbidden to use the short and easy method of the calculus to get your answer. You are bound to get it by the roundabout methods of schoolboy algebra. The Civil Service Commission is responsible for a great deal of the bad teaching in this country. The Civil Service Commission sits as a nightmare on all attempts at spelling reform, and sits on all attempts at mathematical reform. The teaching of arithmetic should follow a certain course, and pupils should be taught how to differentiate and how to integrate simple algebraic expressions before we attempt to teach them geometry and these other complicated things. The dreadful fear of the calculus symbols is entirely broken down in those cases where at the beginning the teaching of the calculus is adopted. Then after the pupil has mastered those symbols you may begin geometry or anything you please. I would also abolish out of the school that thing called geometrical conics. Teach it as a pure piece of geometry, and do not confine it to conics. There is a great deal of superstition about conic sections. The student should be taught the symbols of the calculus and the simple use of these symbols at the earliest age, instead of these being left over until he has gone to the College or University.

MRS. SHAW: It may appear presumption in me to speak in this assembly, but I should like to be allowed to say a few words

on the teaching of mathematics in girls' schools. The Universities have decided, through their Local and other Examinations, that mathematics shall find a place in the curricula of girls' schools; and since girls are not at present trained as engineers and do not all aspire to be pure mathematicians, we may ask what special educational object is sought to be attained by the study of this subject. I think that most people will agree that the object in the case of girls is to develop those powers in which the feminine mind is said to be peculiarly deficient—the powers of accurate observation and logical reasoning. With this end in view it is somewhat disappointing to find that the majority of girls dislike their lessons in such elementary portions of the subject as arithmetic, algebra, and Euclid. The mind cannot be trained by the pursuit of a subject that is distasteful to it, the very distaste is sufficient evidence that the subject is not understood, and therefore cannot be assimilated.

We are, I suppose, all agreed that distaste for the subject of mathematics is induced by erroneous methods of presenting it to the immature mind; though we may differ, according to our several capacities, in what seems to us to be a useful and attractive form of teaching.

I think the mistake has been that teachers of mathematics coming from the Universities, filled with ardour for the subject and veneration for their lectures and tutors or for some other reason, have sought to interest children by methods suited to young men and women between nineteen and twenty-four years of age.

The Universities themselves are not innocent in this matter. The Examination papers for children differ only from those of the Highest Honours Examination in that the questions contained in them cover a smaller portion of the subject. Euclid's theorems appear in papers for students of all ages, the more valuable theorems with longer proofs being set to the older students, while those of little value but having shorter proofs are set to the children. The work of preparation is of the same character for all.

Supposing that in teaching we divide the study of geometry into two parts, the first part being the development of such abstractions as a straight line, a plane surface, and different geometrical figures, none of which are ever perfectly realised in the concrete; and the second part deductive reasoning about these abstract notions; we find examination papers even for the youngest children devoted

exclusively to the second part of the subject, and consisting entirely of questions which may be answered either rationally or from memory, the words used being identical in either case; we find no questions to serve as an encouragement to spend time and trouble over the first part of the subject, a part which is nevertheless essential to a rational answer in the second part as distinguished from an effort of memory.

Out of school if not in school children are constantly learning by observation of the things around them, and they as constantly make inductions from what they observe, they delight in the observation of new objects, but the childish mind knows nothing of abstractions. When we speak of a "clever" man we often mean one who has no difficulty in making correct abstractions, but we know that clever men are scarce, and the majority of children in our schools are not clever, and do not easily make correct abstractions. Our efforts should therefore be directed to assisting them by teaching from concrete familiar objects and facts in such a manner as to prepare the way for the later formation of abstractions.

It has always seemed to me that elementary mathematics could be taught to children by a teacher of science. The science teacher has at his command facts and apparatus with which to illustrate his subject. Arithmetical rules, algebraical signs, geometrical forms can be illustrated by objects familiar to quite young children; boys and girls of ten or eleven years of age take a delight in observing the forms and qualities of material objects.

I do not think that much would be lost educationally by omitting parts of subjects, "skipping" as Professor Perry calls it, if the parts studied were thoroughly understood. It is the pernicious habit of using symbols to save thinking that renders arithmetic and algebra useless as educational subjects, and time would be well spent in reverting again and again to the meaning of the symbols which the student uses to obtain his results. The teacher of science would be able to give life to arithmetical calculations and meaning to definitions by using such simple apparatus as the balance, squared paper, protractors, scales and rules of all kinds, and should in passing to the study of algebra treat it in the first place as symbolical arithmetic, and encourage the pupil to represent graphically his equations and formulæ.

Geometry is perhaps the subject which suffers most from inefficient teaching. I do not agree that we should abolish Euclid from the schools, Euclid's methods are very valuable as specimens of sound reasoning, and as illustrating the nature of a proof; the absurdity of our present system lies in using Euclid as a means of teaching geometrical facts. For that purpose the children should handle solid bodies, and from them obtain plane figures. The faces of crystals supply many of the plane figures treated of in Euclid, and obtaining the figures from natural objects gives life and interest to the subject such as cannot be induced by drawing lines arbitrarily on paper. When the names of the figures are familiar language to the child, and he has for himself observed and classified their properties, he may be interested by an account of Euclid and his times, and be incited to find how the properties he has observed for the few objects he can handle can be shown to be generalised truths for abstract forms. Moreover, it should not be forgotten that mathematical subjects have also a long and interesting history, and a great deal would be gained from a literary point of view if the history of mathematics to the part reached by the pupil were made known in a reasonable manner. The lives of great mathematicians and other men of science should be studied, the state of the special subject when they lived should be known, and their additions to it made manifest. It could not fail to be stimulating to the best of students to learn something of what any man of genius had done to extend our knowledge of his subject, and the benefits we reap from such extension.

The difficulty about new methods of teaching is that we have neither Examiners able to point the way nor teachers fitted to carry them out. It is not sufficient to take the ordinary teachers, whose educational experience is limited to the conventional methods, and by a short course of lectures arouse in them a temporary enthusiasm for the new methods. I have myself tried that plan. The enthusiasm was there and all seemed intelligence and interest until I followed the teachers into the schools and heard the reproduction and extension of what I had taught. In many cases the patching of the new cloth into the old garment was a lesson in more senses than one. The work of training teachers in new methods must be a long and slow process,

involving as it does going behind the conventional knowledge of a text-book that is at present tested by examination, and developing the fundamental principles and ideas of the subject. How many students of arithmetic for example have any idea of selecting a scale as the basis of numeration or of estimating a scrap of paper as a fraction of a sheet?

The Universities supply the Examiners, and the method of Examination is a tradition of long ages. The subject of Examination, and by implication the method of teaching, is defined by a short paragraph of a syllabus or by the name of a text-book, and the Examiner's report is strictly limited to the relative achievements of the children with occasional remarks on the quality of the teaching.

If new methods of teaching are to prevail the Examiners must examine on new lines. It has been urged against the more rational methods of teaching that the information imparted does not admit of examination, but this is not the case, and even if it were it would be better to devise some substitute for the examination, for the teaching is the more important matter from the point of view of education.

To efficiently examine on the new lines we should require more Examiners than are employed at present, and each Examiner might have to devote more time to the work and therefore require to be more highly paid. It would be the duty of the Examiner on his appointment to visit the school and see the class at work. To learn from the teacher what he had endeavoured to teach the children, to find by oral questioning what appreciation the children had for their work, and finally to frame a paper of questions so worded that the children would have no difficulty in at once understanding each question. His report would point out where he thought the results fell short of the teacher's aim, and would contain suggestions as to the manner in which the teacher might with more certainty attain his end in the future.

PROFESSOR GREENHILL: Leaving generalities aside, I would direct attention to the book *Cours d'Algèbre conforme aux derniers Programmes* (Poussielgue, Paris, price five francs), as showing the superiority of French methods over our own, and the backward state of our mathematical instruction.

The French schoolboy is there initiated at an early stage to the notation and elementary conceptions of the *Calculus* in the course of a chapter in *Algebra.*

So also the ideas of *Co-ordinate Geometry* and the plotting of graphs of simple algebraical curves on squared paper are inculcated, exactly as Professor Perry has been endeavouring to effect in this country.

I would sweep away, if I had the power, the study of *Analytical Conics* as a separate subject. It blocks the way of the student to the *Calculus.*

It will be noticed that in Scholarship examination little or no *Calculus* is required; the great important subject is *Analytical Conics,* treated by old-fashioned, calculus-dodging methods.

MR. PARKER SMITH: It has often seemed to me extraordinary that, with the enormous amount that parents and the country are willing to spend upon education, both parents and the country should take so little charge of what is done as the result of the expenditure. It astounds me that children are sent to school and kept at school for a long period, but with no choice exercised and no charge taken by the parent as to how that period is employed, in the faith that what is done there will be for their good. The pressure for examination does not come from the outside; it comes from inside, either from lazy persons in the teaching profession or still more from those who have to test the result of teaching and who will only do so according to fixed and pedantic rules. I think it is useless to inveigh against examinations. We must have examinations; but I see no reason for not going so far as we can in making the subject-matter of that necessary evil, the examination, reasonable, and for not dealing with what is a pure waste of time for the future. It is astonishing the waste of time we give in education. Take the amount of time wasted in spelling, and the system of having separate tables for everything with unreasonable multiplicands. With a simpler system of spelling and a metrical system for our weights and measures, what an enormous weight would be lifted from the rising generation! Truly he would be a benefactor who would establish such a reform, and when that ad been effected we might really have time to turn to the more easonable and more interesting teaching of mathematics.

Professor Olaus Henrici: I agree that the method of teaching should be developed by the student himself on the concrete system. I also believe that the whole teaching of mathematics would be revolutionised if every child in the country were taught something of geometrical drawing. The elements must be taught scientifically; the child must understand the reasoning and the meaning of the result of reasoning. There should be not merely drawing to scale but the drawing of the same figure to different scales. Then the sixth book of Euclid becomes axiomatic. I should like to see these drawing exercises extended far beyond what Professor Forsyth suggested. Then as to skipping, that can only be done by skilful hands. I hope mathematicians will not find it beyond their dignity to write more elementary books on the subject in a more scientific way.

Mrs. Nathaniel Louis Cohen: What I would like particularly to know is, who is to begin this new theory of teaching? Who is to teach the parents? Who is to teach the teachers and examiners? Allusion has been made to the pathetically confiding (and somewhat ignorant) way in which parents send their children to school, leaving the how and what of the learning entirely to the schoolmaster.

The preparatory schoolmaster stands or falls by his success in passing boys into the public schools, which in their turn win their laurels by success in gaining scholarships at the universities. On the universities, therefore, lies the onus of the reform. They are the ultimate deciding force in shaping the grooves of education. I believe the examination for the public services is largely influenced by the universities too, and that the examination demon might be exorcised by a bold show of originality and practicality at the accredited and time-honoured fountain-head of secondary education.

Sir John Gorst in his presidential address to the Educational Section of the British Association expressly disclaimed paramount State guidance in national education, and ascribed to universities and public schools a large measure of influence in shaping public opinion on education.

It has been noticed that the cut-and-dried forms of mathematical knowledge are extremely difficult for some minds to grasp at all

but if they are linked to tangible illustration they seem to find a hold in the mind.

This circumstance has prompted the Education Committee of Queen's College, London (the oldest of the women's colleges), to organise a course of Domestic Science, embracing the chemistry of food and of laundry work, &c., &c.—with practical teaching in cooking and laundry work—and a course of physics exemplified in hygiene, sanitation, warming, and ventilation. Courses in chemistry and physics shaped on these lines are already started, and the full course will, we hope, be started next year.

PROFESSOR ALFRED LODGE: I wish to urge, with Professor Perry, that the teaching of elementary mathematics should be as far as possible concrete; that, for example, areas of rectangles and other plane figures should be calculated from actual measurements made by the pupils themselves, and that volumes and surfaces of solids should be studied in the same way. The distinction between lengths, areas, and volumes would by such means become instinctive. The following wonderful answer with reference to the length of a walk along specified lines in a square 10-acre field would then be impossible: "If the man walks from A to B, from B to C, from C to D, and from D back to A, he walks 10 acres, and as there are 640 square miles in an acre, he walks 6400 square miles."

Pupils should be trained also in approximate calculations.

This work can be made to help other subjects; for example, the number of square miles in any country or county can be quite easily calculated by the pupils by using tracings and dividing them into strips or squares by lines ruled at convenient distances to facilitate the calculation. In connection with practical calculation it should be easy to train the pupil to multiply and divide by contracted methods, and to avoid stating a result to a greater number of figures than the data warrant.

In connection with formal algebra, the distributed product of $(a + b)$ $(c + d)$ can be made very real by computing areas of rectangles whose sides are given in feet and inches. In such ways the pupil may be made to feel the reality of the processes he is learning, and be interested in them, even before he arrives at the solution of equations. In other parts of the subject there

are of course many ways in which geometry (plotting of variable quantities, solution of simultaneous equations, &c.) can be made helpful and stimulating. Algebraical processes, simplifications, factors, &c., should occasionally be tested by numerical substitution.

In connection with geometry many teachers agree that Euclid as a text-book labours under many disadvantages; it pays more attention to logic than geometry, and the constructions given in it are for the most part eminently unpractical. It is too much divorced from practical geometry and mensuration. But it is easier to find fault than to suggest a remedy, especially so long as a knowledge of Euclid is absolutely necessary for examinations. But even in the study of Euclid much could be done to make its results more real and interesting to the average boy. Some practical acquaintance with geometrical magnitudes and measurements should precede any study of Euclid. Then in working at Euclid accurate but varied figures should be insisted on, and results should be frequently tested by actual measurements and numerical work. Many ideas become thoroughly assimilated "through the fingers" which would otherwise remain in the air.

Examining bodies could materially assist in the more rapid assimilation of Euclid if they would agree to a few modifications of their system of setting questions; for example, by agreeing to always omit I. 2 and 3 as being unpractical; by always setting a proposition *and* its converse where the proofs are connected, instead of divorcing them and often setting the converse only, the proof of which is but half a proof; by relaxing the rule of "Euclid's order" as regards "constructions," *e.g.* allowing I. 5 to be proved (as has been prettily done in an American book) by imagining the vertical angle to be bisected, thus dividing the triangle into two parts to which I. 4 can be at once applied. Problem work would receive a great stimulus if examining bodies would set *separate* problem papers, instead of being content to insert riders in a book-work paper, which practically compels even clever boys to leave them till they have answered all the book-work.

As regards some modifications of Euclid which seem to be desirable: I. 2, 3 should be cut out, and replaced by a postulate. I. 45 should be entirely rewritten, so as to bring it into accordance

with the methods of the drawing office. I don't know whether any one has ever really tried to use the method suggested by Euclid; it would be most instructive!

I believe the theory of parallels would be much simplified if, instead of being defined *negatively*, they were defined as lines (in a plane) making equal angles with a transversal, on the same side; a definition which is suggested by the practical method of drawing them with two set-squares. The fact that they do not meet would then follow as a deduction from I. 16. The twelfth axiom would still be required as the converse of I 17, but only in connection with such propositions as I. 44, to justify a construction.

Euclid II. should be definitely made an introduction to the *algebraic* treatment of areas, and be closely associated with mensuration. Algebraic methods (and arithmetical ideas) should be cultivated as on the Continent, instead of being looked at askance as being non-Euclidean. This would pave the way to the practical treatment of ratio in Book VI.

I have mentioned only a few points in which I think improvement could be effected, pending the introduction of a standard geometry other than Euclid or the abolition of a standard if that is possible with so many public examinations. I may add that I am entirely in sympathy with Professor Perry in his desire that mathematical work should be made more practical and tangible, so long as it is not less thorough.

PROFESSOR MIALL said a few words about the teaching of elementary geometry. A particular elaborate and artificial system, hampered by conditions too strict to be always observed, conditions whose purpose is unintelligible to most young minds, is regularly imposed upon all English pupils. The beginner can neither construct nor apply such a system for himself; he can only "get up" what he finds in the book. When we object to Euclid's Elements as wholly unsuited to the wants of the pupil, we are always asked what text-book we can propose in its place. Useful books can easily be named (a few were shortly noticed), but why are we so solicitous to have one recognised geometry, any more than one arithmetic or one trigonometry? Why should not every enterprising teacher frame his own course, as he often

does in Germany or Switzerland? Then we are asked: How can the pupils be examined in geometry if one has studied one book and another a different one? What a measure does that question give of our educational methods! First comes the examiner; then, I suppose, the teacher; but where does the pupil come in? The difficulty does not exist outside the English-speaking nations, who have Euclid as a school-book entirely to themselves. Elsewhere it is not found to be too hard to say who is competent in geometry and who is not. The teacher in all parts of his work is too much dominated by examiners and inspectors; he is not at liberty to try things for himself. The detailed syllabus, prescribing minutely what he is to do, and how he is to do it, inevitably enslaves the teacher. He ought to be able to work on lines of his own. Occasional examination may be useful to show what his pupils can do, but the paper should never require knowledge of a particular book or a particular method. No examination can be altogether satisfactory in which the teacher does not take an active and responsible part.

Professor Minchin: There can be no satisfactory progress in the teaching of mathematics in this country until Euclid is got rid of. The Civil Service Commissioners have been blamed, but I think unjustly. The Civil Service Commissioners have made as large a concession as it is possible for them to make in regard to the teaching of Euclid. They have said that they no longer require from candidates Euclid's demonstrations. What they ask is that there should be no departure from Euclid's order. I think Euclid's order is bad and should be departed from, but I do not see what else the Civil Service Commissioners can do because the real culprits in the matter are the head-masters of the public schools who stand in the way, and will have Euclid as the text-book. I think that geometry ought to be taught to the young in connection with arithmetic. Again, I think that those who teach mathematics have not yet learned the enormous value of graphical methods in the solution of all kinds of problems.

I also think that our mathematical books—particularly those on applied mathematics—leave much to be desired in one important respect. They suffer from a lack of imagination on the part of the authors. The average English author leaves one under the

impression that he has made a bargain with his reader to put before him the truth, the greater part of the truth, and nothing but the truth; and that if he has put the facts of the subject into his book, however difficult it may be to unearth them, he has fulfilled his contract with his reader. This is a very much mistaken view, because *effective teaching* requires a great deal more than a bare recitation of facts, even if these are duly set forth in logical order—as in English books they often are not. The probable difficulties which will occur to the student, the objections which the intelligent student will naturally and necessarily raise to some statement of fact or theory—these things our authors seldom or never notice, and yet a recognition and anticipation of them by the author would be often of priceless value to the student. Again, a touch of *humour* (strange as the contention may seem) in mathematical works is not only possible with perfect propriety, but very helpful; and I could give instances of this even from the pure mathematics of Salmon and the physics of Clerk Maxwell.

I think also the viciousness of examinations, of which complaint is often and justly made, would be greatly obviated if the teachers were taken into confidence. I think that in the direction that has been laid down by Professor Miall lies the true solution of this great difficulty.

PROFESSOR ANDREW JAMIESON: As an engineer, electrician, technological teacher, and author of engineering books, I most heartily agree with Professor Perry's pithily expressed views and excellent "Syllabus."

I was born in the northern county of Banffshire, where the finest and best set of parish schoolmasters in the Kingdom were to be found. Their pre-eminence was no doubt largely due to the beneficial effects of the "Dick" and the "Murray" bequests. These "mortifications" enabled schoolmasters who held the Master of Arts degree from a University, and who had also passed other special examinations, to receive substantial additions to their salaries. In those days there were no cramping, hard and fast "Codes." Each teacher was not only stimulated by the keen "Bursary Competition" at Aberdeen University, to produce better students than his colleagues in the neighbouring parishes

and counties, but also, if possible, to surpass the results of the Grammar Schools.

Moreover, the work of the schoolmaster was greatly encouraged by the clergy, in their private capacity as "ministers" of their respective parishes, and in their collective "presbyterial inspections." There was no waiting for formal printed reports. If a certain student or a class answered well, then out came the spontaneous praise, or, perhaps, the voluntary present! The schoolmaster and the clergymen gathered in a friendly way at the Manse, and the scholars went home together after the inspection, each to discuss in their own way the results and honours or failures of the examination.

Further, in those days there were excellent teachers of mathematics and natural philosophy at Aberdeen University. At Marischal College, there was the famous scientist, Clerk Maxwell, and at King's College, there were the well-known mathematician, Professor Frederick Fuller, and the best of natural philosophy teachers, Professor David Thomson. Their influence spread far and wide. The latter, up to the very time of his retiral, systematically brought his students to the black-board and made them demonstrate before the whole class what they knew—or what they did not know—of the fundamental principles and the applications of physics. Woe betide the student who made a fool of himself, for his fellow students watched with eager eyes and ears for any slip or idiosyncrasy, and were not slow at expressing their approval or disapproval. In this way, good mathematical teachers were trained, encouraged, and spread throughout the country. Besides which, the many Wranglers at Cambridge (amongst whom were an extraordinary number of Senior and Second Wranglers, as well as "Smith" Prizemen) who emanated from the North of Scotland, raised an *esprit de corps* amongst the teachers and taught.

In the parish schools some forty years ago, boys and girls were taught together. I remember commencing Latin with six farmers' daughters! Many a "tift" took place between the opposite sexes, to see which would first reach and keep the "top."

"Common-sense and compromise" existed in a free and easy way. Whenever a teacher thought fit, he would order a class to retire to their part of the school, and discuss their lessons

together; in fact, "to think out the subject for themselves," until *he* was ready to hear them. There was a minimum of home lessons. Practical illustrations from nature were freely taken advantage of. Mere memory work was discouraged. Affection, as distinct from fear or distrust of the teacher, was cultivated. The eye, ear, touch, and even smell were all educated. For example, boys and girls would be asked to write down upon their respective slates, what each one estimated the length of a rule, desk, or room to be. The slates were then turned. The article was duly measured, and the one who had produced the most correct answer passed to the "top." In the same way, areas, volumes, and weights were dealt with.

In after life I often blessed that schoolmaster. For, when out repairing Submarine Cables, I was able to save much time and trouble, by quickly estimating the distance of "mark buoys," lighthouses, and headlands; and in the workshop or laboratory, sizes and weights, &c. At the same time, I have often regretted the many hours misspent in the Secondary School and College, grinding up the old-fashioned system of Euclid, and in trying to solve unpractical and uninteresting mathematical questions.

I am sure that we all owe Professor Perry a deep debt of gratitude for bringing to a focus, and so earnestly requesting our several opinions upon, his rational system of teaching mathematics.

W. R. Cooper: Not being a teacher I cannot speak from the teacher's point of view; but judging from my own experience of being taught I think that Professor Perry is trying to get more into a given time than can be done efficiently, with the result that the state of hurry, which is already excessive for the *average* mind, would be too great. Present-day education is apt to be disappointing, because the student has not sufficient time to think round his subjects and the possible difficulties that may be in them. Probably the only way to remedy this state of things, which is more or less equivalent to "cramming," is to introduce research work, or work of that character, as soon as possible, for it is only by research work that a true knowledge of a subject can be gained. Moreover, routine work is likely to kill initiative. But it is difficult to determine how soon research work can be profitably undertaken, and it probably means more time.

Sir John Gorst: Strong language has been used by some of the speakers, but it has been used in extremely good humour, and it has very much enlivened our proceedings, and I hope it has brought home the meaning of the speakers in a very emphatic and clear manner. I should have liked very much, as an old mathematical teacher both in schools and colleges in England, to have made a few remarks upon the general question had time permitted (applause). I am encouraged by the applause of my audience to say one or two words (applause). In the first place I should like to say how extremely useful a practical discussion like this is. If all the mathematical teachers in England could have been present to listen to it, or be brought to read the account of the discussion which Professor Perry is going to publish, I am quite certain the mathematical instruction of boys and girls in this country would be at once enormously improved. I only regret that so many of our speakers have been professors in universities and so few of them have been practical teachers in schools. In teaching mathematics in my younger days I had what was a very unique experience. I taught, or attempted to teach, mathematics to the Maori boys and men in New Zealand. Now as far as the teaching of arithmetic went I taught on a sort of embryo Sonnenschein principle, and I found them remarkably apt and quick pupils. They learned the practical arithmetic, which was useful to them in actual life, and they learned it with extraordinary rapidity—far faster than boys or men would generally learn it in this country. But when in my youthful enthusiasm, finding the extreme rapidity with which they learnt the rules and practised the problems of arithmetic, I proceeded to try to teach some of them Euclid, or rather geometry after the Euclid fashion, I absolutely and entirely failed. There was not one of them that could grasp or understand the simplest of the propositions of Euclid. I suppose the reason was because it was impossible to present that aspect of mathematics to them at that time by such a teacher as they then had (laughter) in a concrete form. Had I had the advantage of the discussion to which I have listened to-day, I should have abandoned teaching in the ordinary way until they had been familiarised with angles, lines, areas, and geometrical figures, of which the Maori youth was absolutely ignorant. I suppose by a method of that kind even

the least developed intellect of the uncivilised native of New Zealand might have been brought to take in some of the very simple propositions of geometry. As it was, proceeding in the fashion in which one did proceed in those days, the attempt was an absolute and total failure; and I think it rather confirms the general opinion which has fallen from this section in the discussion this morning. It might have been possible to make Maories familiar with the concrete objects of geometry, but it was absolutely impossible to get into their minds those general and abstract propositions which to them had no meaning whatever (applause). That is really all the contribution I think I can make to the discussion (applause).

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In answer to request the following remarks were communicated :—

Principal Oliver Lodge: As I was not able to be present at Glasgow, Professor Perry has asked me to add a word or two to the discussion which he aroused there. I hope before long to have something to say more at leisure and more at length on the subject, which I regard as one of extreme interest and importance. It is a subject which reformers have attacked from time to time with vigour, but none of them so energetically or hopefully as Professor Perry is attacking it now. I think that most people agree that some reform in mathematical teaching is desirable, and many teachers have already on their own account introduced some such reform or some portions of such reform as is now advocated.

The introduction is however by no means general, or rather it is peculiarly individual, local, and exceptional, and until the great examining bodies take the matter up it will be very difficult for schoolmasters to teach in the way many of them would like.

What I am most impressed with is that the beginning of mathematics should be taught by people who know a good deal about its broader features and its applications, and have some enthusiasm for them. Naturally everybody knows the beginnings of mathematics after a fashion, but it is a great mistake to let people who know no more than this teach what they know.

I am not acquainted with any other subject where a wide knowledge on the part of the teacher is equally important; because without it the subject is dull and depressing, whereas with it, it can be made to bristle with life and interest and illustration. It is like the difference to a child between a school-book and a coloured picture-book. And I quite agree with Professor Perry, in substance if not in detail, that in all probability a large proportion of children, if properly taught, might like mathematics, and proceed in it a very considerable distance, instead of hating it and becoming blocked at the threshold. There are details in his statement, however, with which I am unable to agree, but, after all, my difference with him is rather one of emphasis and proportion. He appears to press the impatient exclamation of Sir Peter Teazle, which he quotes on page 21, into sober earnest, emphasising the importance of facts divorced from principles in a somewhat exaggerated manner. But I expect that he would reply that all this is for the sake of making his position clear, that all he says has to be read with a grain of common-sense, and that the changes in the direction he indicates are so desirable that perhaps he has appeared to push their advocacy a little too far.

At any rate, I feel that geometrical propositions unproven or assumed are valueless, becoming no better than engineering pocket-book formulæ which can be used by a so-called practical man who does not understand the reasoning by which they were arrived at. He may get correct results by their use, but it is a cramped and unnatural method, and he is extremely liable to make mistakes. It is not undemonstrated or assumed propositions, therefore, that I would advocate, nor that Professor Perry would really wish to advocate, but propositions demonstrated again and again by every sort of method, till their place in the whole scheme of things was perceived, appreciated, and realised. It is this realisation of the way a fact fits into its place, being proven on all sides and not only on one side, that constitutes real and genuine demonstration, of a kind that gives absolute confidence. The "linear" demonstration of Euclid, where in order to be rigorous the whole order of the propositions must be remembered, is artificial; and although supposed to be peculiarly crucial it is not really demonstrative in a satisfying manner,

because it is easy with a little ingenuity to demonstrate a fallacy by such a process as that; and it takes more than a beginner to be able to detect the flaw. Wrong-headed people of no mean ability, such as have been studied by Professor De Morgan constantly deceive themselves in this manner, and a schoolboy who was taught to put his faith in a linear demonstration or chain of proofs might easily fall a victim to this disease; and even if he did not, it is injudicious for him to imagine that he could always reason himself into the truth by a strictly Euclidean process, and that he would always be able to detect the flaw when there was one.

I venture to say that the Euclidean geometry considered as a philosophical system, though highly ingenious and delightful from some points of view, is based upon a fallacy—that is to say, is based upon an erroneous view of what constitutes proof, an erroneous view as to the nature of axioms: a view which has largely obtained, among ordinary men even if not among philosophers, and a view which is responsible for a great deal of fog and uncertainty about the fundamental nature of our knowledge, *e.g.* for the controversy between intuition and experience.

The basis of experience underlying all axioms is so purposely masked in Greek geometry that it has been thought (and I suppose by Euclid was thought) to be non-existent. And so geometry appears to be built upon air—a kind of mental figment, a self-woven wraith, instead of what it really is, a very abstract variety of science; and as such it has constituted a bad introduction to Science, and so far from assisting to withstand a tendency to deal with book knowledge alone, it has over-emphasised that tendency, and polluted what might have been the earliest effort of the mind to get into reasoning contact with Nature herself.

Oliver Heaviside: I have read your Address. I do not doubt that in matters of detail, in practical application, many points of divergence might arise between us. For example, I may not think so much of the slide-rule as you do. And as regards squared paper, what I object to is that the lines are not equidistant, and that is an offence to the artistic eye, besides introducing visible error. But these and similar matters

are trifles in view of the most important object you are aiming at. It is fortunate that we possess an educational reformer who is so earnest, enthusiastic, and persistent, and I hope you will meet with your reward in some substantial realisation of your ideas. So far as I can see, there is a large body of educationalists in fair agreement with you at present. These have been, for long past, isolated would-be reformers. What is, I suppose, the principal difficulty, is the stupid old Toryism of the leading Universities. But even that has partly given way. If it is any satisfaction to you to be told what you know, I beg to say that I very much approve the general spirit of your Address. I agree entirely that mathematical works for the instruction of boys, sometimes including big boys, maybe up to twenty-five years of age, are generally written on wrong principles. Boys are not philosophers and logicians. Boys are usually exceedingly stupid in anything requiring concentrated reasoning. It is not in the nature of their soft brains that they should take kindly to Euclid and other stuff of that logic-chopping kind. But they usually possess another sort of mental ability—namely, the ready acquisition of new facts and ideas—and that is what should be taken advantage of. They have also the power of learning to work processes, long before their brains have acquired the power of understanding (more or less) the scholastic logic of what they are doing. I have known boys of fourteen extract cubes, fourth and fifth roots of numbers, to several places of decimals, easily. I am sure most of them would never live to understand the reason why, if they studied for a thousand years. Now, the prevalent idea of mathematical works is that you must understand the reason why first, before you proceed to practice. This is fudge and fiddlesticks. I speak with confidence in this matter, not merely from experience as a boy myself, and from knowledge of other boys, but as a grown man who has had some practice in applications of mathematics. I know mathematical processes, that I have used with success for a very long time, of which neither I nor any one else understands the scholastic logic. I have grown into them, and so understand them that way. Facts are facts, even though you do not see your way to a complete theory of them. And no *complete* theory is possible. There is always something wanting, no matter how logical people may pretend.

The fact is, there is theory and practice in mathematics just as in everything else. A man may be a good musician, by practice, and yet know nothing about the theory of music. He may arrive at that, or he may not. The same applies all round, and mathematics is no exception. The theory of mathematics is very important, but it is not the same thing as its practice, and the important matter here in relation to young persons is that the practice should come first. Then familiarity may gradually lead some to understanding of the theory, which may be studied later on in a rigorous manner, if the developing boy should be mentally fit for close reasoning. This is particularly true, I think, in geometry. It should be entirely observational and experimental at first, a natural continuation of the everyday education acquired through the senses. And even when you do come to the theory, you should put aside all old-Toryish ideas, and logical tricks and puzzles, and let the boy work more practically with the assistance of arithmetic and algebra.

Now the majority of British boys are not fit for any sort of logical theory. It is mere waste of time forcing them through Euclid, just as it is waste of time forcing Latin into them. The little they learn is soon forgotten, and gladly. What is it done for? I see nothing but old Toryism in the common arguments. Did the Romans and Greeks make their children painfully study scraps of ancient dead languages?

There is so much to learn nowadays, really valuable knowledge of all sorts, that it seems to me a wicked sin to go on in the old way, with Latin, Greek, and Euclid. It takes such a time, involves such labour, and does so little good. And we are living in the twentieth century.

The general inability of boys to study Euclid profitably is no reason why they should not learn geometry. Even stupid boys can do that, when properly directed and experimentally assisted. And as regards mathematics in general, I think it a very important assistance to have it taught in conjunction with elementary physics. That is, geometry *and* other natural facts.

As regards clever boys, I agree with you that it will do them good to go through the same process in the first place.

I seem to be running down logic. I do not mean to. But there is logic *and* logic. There is narrow-minded logic confined

within narrow limits, rather conceited, and professing to be very exact, with absolutely certain premisses. And there is a broader sort of logic, more common-sensical, wider in its premisses, with less pretension to exactness, and more allowance for human error, and more room for growth.

LORD KELVIN: Many thanks for your letter of yesterday and copy of your Glasgow address, which I have read with interest. I am overdone with work which must not be postponed, and I am sorry therefore not to be able to write anything on the subject. I think your syllabus very good indeed. It is very like the teaching I had from my father.

DR. J. LARMOR: I believe the methods advocated by Professor Perry are in the main the right ones for beginners, and are in fact the methods actually followed or aimed at by most reflecting people who have to do with elementary education whether in school or college. The difficulty in regard to them is, I think, a practical one. There is the question of expense: to carry out such a plan involves small classes and much individual attention to dull pupils, whereas the present more routine methods have been evolved largely in order to enable a single teacher to supervise a large class. There is the difficulty of obtaining a sufficient supply of elementary teachers who could be trusted to pursue a more unfettered course without exaggeration or wrongheadedness. On both these grounds I fear that the results of putting such a plan into general operation under present conditions might be disappointing; anyhow they are not to be inferred from what can be done under specially favourable circumstances with picked pupils, often of mature age and with the acquired instincts of workshop practice.

I believe the outcry against examinations is a mistake. At various times I have observed what a persistent examiner can do in the course of a year or two to modify, on a very extensive scale, modes of teaching which he considers to be defective. Therefore I believe that there is no way in which the best educational talent in the country can be better employed than in the performance of the duties of public examiners. Once it was so recognised: but now the occupation has been so much discredited that competent persons are averse to taking it up. It is too customary to

consider that any one with some spare time is good enough to be an examiner; and no fee is too small to offer him, seeing that if he declines there are sure to be others ready enough to grasp at it. Examinations can be of great utility to direct the course of instruction where supervision is needed or helpful; at present I believe their stimulus and guidance are (or could be made) helpful in all places of elementary instruction except some of the higher schools and colleges. Moreover, education is largely the learning to arrange and express one's thoughts; and in this sense answering intelligent examination questions is as educative as anything else. I am even of opinion that if there is to be a reform in the methods of elementary instruction, the most promising way to work towards it is through carefully conducted public examinations.

W. N. Shaw: I had no intention of taking part in this discussion, because among other reasons one educational authority in a single household would seem to be sufficient; but in response to Professor Perry's request I will venture to add a few words based upon many years' experience, not as a teacher of mathematics, but as a college tutor and lecturer to students for whom a knowledge of mathematics is, in various ways, a necessity.

In my Cambridge days a college tutor was not always regarded as a person likely to be interested in school education; the College Scholarship examinations might easily account for such an impression; but in watching, with a certain amount of intelligence, and as far as was practicable assisting, the ingenuous youth from the end of his school days to the beginning of his real life, I found it impossible not to ponder over the apparent results of some ten years of persistent and expensive educational effort. In that way I have come to have an opinion upon the teaching of mathematics in schools and in the university, and the extent of its success.

It is apparent at the outset that mathematics may be taught to three different classes of students for three different purposes,—namely, first, as a part of general education; secondly, as an academic training for general professional life; and thirdly, as a necessary introduction to study for professions in applied sciences. Professor Perry's paper deals with the last of these three purposes. I will deal with it later. I wish to say a few words first about the others.

It used to be my fate to have to continue, with hydrostatics and heat, the study of the elementary mathematics for the purposes of general education that had been represented in the schools by arithmetic, algebra, and Euclid, and possibly a little mechanics. Discussions with successive generations of parents as to the educational purpose of various branches of study, not obviously utilitarian, has led me to conclude that the traditional school study of scientific subjects must have been intended to give the boys an insight into scientific method and practice in scientific reasoning; and, upon proceeding to the continuation of the subject, I used to devote a little time to finding out how far the class had realised this purpose. By way of text I generally gave out the oracular sentence, "Hydrostatics is the science which treats of the equilibrium of forces acting on a fluid." The sentence was forthwith transferred to many note-books with complete fidelity; and I thereupon proceeded to inquire whether my oracular definition had made anything any plainer to anybody by ascertaining what meaning was attached to the words of the explanation. I always found that the word "science" conveyed no definite meaning at all, and the rest of the hour was devoted to discussing, not without difficulty in view of traditional methods of school teaching, how it was that the mathematical subjects they had learned at school were called sciences.

As the term wore on and my acquaintance with successive classes deepened, I had reluctantly to allow, year after year, that it is possible to spend years over the study of elementary mathematics and to pass with credit an examination in Euclid without comprehending the nature of a proof or the elements of scientific method, or an examination in algebra without carrying away even the fundamental algebraical idea of reasoning about a quantity represented by a symbol, or an examination in arithmetic without any more mental equipment than a few rules of thumb. I had also to realise that, although a part of general education is training in the accuracy of statements intended to convey ideas, it was permitted to write down on paper statements that were outrageously and obviously untrue, sometimes palpably absurd, without any shock to the conscience or the sense of taste, unless the graceful rounding off of the "sum" with "o = Q.E.D." (a favourite quotation of an esteemed mathematical colleague) can

be regarded as intended for the propitiation of some offended goddess.

In all seriousness the teaching of mathematics in schools as a part of the general education of the average youth is in my experience a pitiful failure, which results in many cases in the student regarding the subject as merely foreign to his intelligence.

It seems almost as ungenerous as it is common among inexperienced critics to attribute the failure of pupils to faults in the teaching, and if I join with others in attributing the apparent failure of mathematical teaching in schools to the nature of the teaching it is because on looking through accepted text-books of the fundamental sciences I find no echo of the way in which every science is developed, through patient observation, classification, and induction to the deductive study that is essentially mathematical. I find a careful and logical presentment of the later stages in the development of the subject interesting to the experienced student looking back over the ground he has traversed, but separated from the beginner by the whole stage that corresponds with the inductive period in the development of a science.

The teacher and the beginner who is unable to work out for himself the inductive stage move on different planes, and, in consequence, the student learns his science as he would a foreign language, by the use of his memory and his formulæ. He is using the wrong faculties and abhors the study.

I cannot help thinking that this state of things, which has its counterpart in a science student's abhorrence of literary studies, arises ultimately from the specialisation of teaching in schools. It is not, I hope, unfair to say that adequate general education is imperatively required in every profession or business except that of a teacher. A one-sided man cannot be a successful lawyer, or doctor, or clergyman—he must be a teacher—there is nothing else for him. If a student of literature abhors science and all its works we may set him to teach classics, and make up for his deficiency by setting a man of "science" attainments—I will not call him a scientific man—who abhors literature to teach the same boys the necessary science. It is, so far as I know, quite contrary to practice for a master to be asked to hand on to his boys education in the proportions and relations which he himself has found stimulating and effective.

It is no uncommon thing for self-educated men to deplore the want of educational opportunity of their youthful days and to make provision for future generations; yet, when the provision is made, and the education is cut up into separate fragments each administered by its own specialist, it is doubtful whether even the pious founder himself, if he could try it, would not find his educational provision revolting to him.

The specialist coming down directly from the university to teach his subject in a school is very likely to have foreshortened ideas of the position of his subject in the scheme of education. I am afraid I am unable to guess what the ideas of a specialised classic are on taking up the education of boys, and I am hardly less unable to guess what general notions the specialised mathematician sets before himself; but if I hazard the suggestion that, coming from Cambridge, his mind will be turned towards the mathematical tripos (*mutatis mutandis* for any other seat of learning), and the requirements of success therein; and if I add that he realises clearly how, by the re-editing of a text-book, some short cuts to the solution of questions will be made easy, and forgets that upon him lies the responsibility of conveying to the greater number of his boys all the notions they will ever acquire of the precision of mathematical classification, the relentlessness of mathematical reasoning, and the majesty of scientific truth, shall I be judged to exaggerate the case? If an expert in science has not sufficient appreciation of the beauty of literature to help his boys in that direction, and no regard for the advantage and importance of being able to convey ideas as well as owning them, why have him in a school? And if your classic thinks science is—what he found it to be—a waste of time, why let him perpetuate that pernicious view by becoming a stumbling-block in the general education of boys?

Let me pass on to say a word or two about the teaching of mathematics as an academic training for general professional life. It has immense capabilities in that respect. If you consider how much of the effectiveness of an administrator depends upon the capacity for co-ordinating appropriately a number of different ideas, precise accuracy of definition, rigidity of proof, and sustained reasoning, strict in every step, and when you consider what substitutes for these things nine men out of ten without special

training have to put up with, it is clear that the man with a mathematical training has incalculable advantages. But look at the examination questions with which his powers are tested. Sixteen papers with twelve questions in each, all in identically similar form, all testing his powers of analysing or remembering or inventing a proof, in all of which he is allowed to use a conventional language familiar only to his mathematical colleagues; none of the questions are designed to test his knowledge of the evolution of the subject or his power of conveying ideas to people unfamiliar with them. Science has its history and its literature, and exponents of science have differences of style and method that are worthy of examination and study, yet I know of no mathematical examination in which the student is encouraged to think of these things. If I am not mistaken he is allowed to have finished his mental training when he can give, in the latest conventional language, an account of the present state of his subject without reference to the heroes of the past or the possibilities and aims of the future. It may be allowed that it would be difficult in an examination to co-ordinate the marks to be assigned to questions on the history or literature of mathematics with those given for the power of producing, at short notice, a short proof of a recognised proposition or a neat solution of a difficult problem; but does that difficulty exonerate those who direct mathematical studies from the responsibility of encouraging the study of a subject from the three sides, historical, literary, and philosophical, that every subject of human thought possesses? Does it justify them in leaving their finished students imperfectly educated for the want of such "all round" study of their subject?

Regarding the study of mathematics as a necessary preparation for the profession of applied science, my experience with students bears out my conclusion with regard to the teaching of mathematics as a part of general education, namely, that the specialised mathematician does not easily meet the requirements of the student. His interest in the subject may be said to begin where his student's interest ceases. If I understand the matter rightly, the applied science student's business with mathematics is to learn not to be afraid of it; he requires to know enough of it to know what problems can be approached mathematically, and to know it thoroughly enough for him to apply it to the case he

wishes to deal with, and to extend his knowledge when occasion arises. This is no light requirement, but it does not involve a knowledge of the subsidiary details that are the delight of professed mathematicians, and rightly so. New mathematical methods can only as a rule be originated and developed by mathematicians; the student of applied science must be content if he can apply recognised mathematical methods to new varieties of old problems. On this ground I explain what has, I think, been the Cambridge experience, that mathematics for applied science students should be taught by applied science teachers. It seems absurd at first sight that in a university where competent teachers of mathematics abound mathematics should not be taught to students of applied science by mathematicians, but in practice it is not so. It is possible that special training might get over the difficulty, but at present it seems that the only way of estimating truly the mathematical requirements of a student of applied science is to study the applied sciences themselves.

I do not enter into the details of Professor Perry's syllabus; different teachers may have different views, but they will unite in the object of enabling a student to deal mathematically with any problem in applied science that can be solved by recognised mathematical methods.

I would venture also to say that in this branch of the teaching of mathematics encouragement should be given to the study in the original text of the masterpieces of applied mathematical literature. In the reading of many of these the advice and assistance of an experienced teacher are invaluable and indeed indispensable, and the careful study at first hand of the way in which the application of mathematics has been introduced in difficult problems is a special department of great value.

I need not now enter upon a discussion of how such study could be encouraged by examination, and whether a knowledge of the original language of the memoir should be reckoned to the student for righteousness in an examination primarily for science; but here again I would repeat a note that I have already sounded, and say that if we regard general education as an advantage, let there be no hesitation about giving credit for it wherever we find it, even if it be in an examination in science or mathematics.

The notion that in an examination in mathematics language

may be disregarded, and no credit given for anything but the bare skeleton of the mathematical reasoning, is a further step in the process of specialisation which has divided subjects into such separate compartments that the study of one branch is made entirely independent of that of others, in spite of pious wishes to the contrary. I believe this is a concession to the teacher and not to the student, and that many of the real difficulties that Professor Perry's proposal seeks to remove would vanish if the teachers were encouraged to utilise and develop widened intellectual experience and interests.

W. P. WORKMAN: I am afraid that the time at my disposal is utterly insufficient to allow me to do justice to the remarkable syllabus which is before me. Speaking generally I believe that it represents a move in the right direction. But I am not clear that it lays sufficient stress on the training in exact thought which our present system certainly supplies, though possibly, probably even, at excessive cost in time. To take an illustration: Euc. I. 35 is verified by squared paper. A boy will ask, in my opinion he *ought* to ask if he has a scientific spirit, "Why are these parallelograms always equal?" In this case he can be answered, for you may exhibit the parallelograms made up of congruent figures. But when he verifies III. 35 by his graduated ruler and a multiplication sum, as I have been in the habit for years of making my boys do, and then asks the same question, no answer can, I think, be forthcoming. It will be, to some extent the same in other parts. The arithmetic syllabus includes "the meaning of a common logarithm." Does this imply all that it seems to do? Are logarithms to be empirical, or is their meaning to be understood? If the former, then again you will meet the question to which you can give no answer, "Why are these things so?" if the latter, then I am not hopeful of driving into the heads of boys of ten to twelve any real comprehension of fractional and negative indices.

It would be very helpful for teachers if you would state your conceptions of the ages at which the various parts of the syllabus might be begun. Further, one would be glad to know whether the various subjects in it should be taken as far as possible concurrently or in succession.

I am surprised that you should introduce the planimeter into an elementary course. I dare not trust my lower classes with such an instrument. Of your five methods for areas I should confine the elementary course to (4) and (5).

I am convinced that you will need better teachers for the new syllabus than for the one which it is meant to displace. Indeed, I am afraid that for a generation this will render it almost impossible for secondary schools to take the matter up with any completeness. A good mathematician may be trusted to teach this syllabus, and to get more out of it than he did from the older, because it will evoke far more interest. An indifferent mathematician would do much harm, because so much is left to his own badly trained initiative. This suggests a little doubt whether the syllabus is itself the best training for its future teachers. Only experience can decide.

Secondary schools are further hampered by the universities; you will have to get the local examiners, the Joint Board, the College of Preceptors, and the University of London *all* to agree to issue an alternative "practical mathematics" syllabus before secondary schools can safely move.

DR. SUMPNER: It may be useful to draw attention to the experience of evening schools in relation to this subject. These schools have found it necessary to supplement their mathematical classes working for the examinations of the Board of Education by other classes, which experience has shown to be needed by students, and to be more applicable to their work. In the Government examinations the syllabus is divided into seven stages and is arranged on orthodox lines. For students going through a complete course of mathematics, or preparing for university examinations, the course is all that can be desired. But the great bulk of evening students are not of this type; a few are preparing for Whitworth scholarships, and for these the examinations are a necessary stepping-stone. There are, of course, besides, some students who gauge their ability by the number of certificates they possess, but the great majority of evening students require mathematics because of its application to some other subject in which they are directly interested. For such students the arrangement of study outlined in the Board of

Education examinations is utterly unsuited. Hence, long before Professor Perry had outlined his syllabus in practical mathematics, and, therefore, long before it was possible for evening schools to gain any grants for teaching such a subject, many of the larger technical schools, such as those at Glasgow, Manchester, and Birmingham, had already supplemented what may be called the orthodox classes in mathematics by other classes in which mathematics of the type outlined in Professor Perry's syllabus was taught. These classes have always been much more popular than the orthodox ones, these latter being very poorly attended, except in the lowest stages. The former classes have been instrumental in attracting students who would not otherwise have been found in mathematical classes at all. It is, therefore, to be hoped that these classes in practical mathematics will be further developed in evening schools, and that something of the same kind will be introduced into the elementary schools, from which the bulk of the students in evening technical schools originally come.

I have no doubt myself that mathematical beginners, whether boys or adults, should go through a course of practical mathematics before studying the subject in what may be called the orthodox manner; if this were done, the result, I believe, would be that orthodox classes in colleges and technical schools would be much increased and not diminished, since a much greater number of students would in this way be attracted to mathematical studies. For evening students, at all events, the existing examinations and text-books are defective; too much time is spent in driving students through exercises needing more or less dexterous manipulation, and too little in showing how simpler exercises of the same type are applicable to problems in every-day life. The subject of the calculus is postponed so long from such courses that the student rarely reaches it. The calculus is the most directly applicable of all sections of mathematics, but the existing text-books on the subject are unsatisfactory, since as a rule they cover in a couple of chapters results which have numerous applications of immense importance, while instead of explaining some of these applications in later chapters the higher developments of the subject are at once proceeded with.

As regards the necessity for the reform of geometrical teaching,

which has already been alluded to in this discussion, I am largely in sympathy with the view of Oliver Heaviside, who has said (*Electro-Mag. Theory*, Vol. I. p. 148):—

"As to the need of improvement there can be no question whilst the reign of Euclid continues. My own idea of a useful course is to begin with arithmetic, and then, not Euclid, but algebra. Next, not Euclid, but practical geometry, solid as well as plane; not demonstrations, but to make acquaintance. Then, not Euclid, but elementary vectors, conjoined with algebra, and applied to geometry. Addition first; then the scalar product. This covers a large ground. When more advanced, bring in the vector product. Elementary calculus should go on simultaneously, and come into the vector algebraic geometry after a bit. Euclid might be an extra course for learned men, like Homer. But Euclid for children is barbarous."

W. D. EGGAR: I did not hear of the alteration o. the date of your paper at the British Association, and so to my great regret I missed hearing it and the discussion which followed. As you have kindly asked for my views on your proposed syllabus, I should like to say that I am entirely in sympathy with its aims. In criticising its details, I will confine myself to the elementary course. Partial differential equations rarely come within my ken, and I doubt if I should find much use for them even under a reformed code.

You speak of our present course as a net to catch the man in ten million, the great pure mathematician. My experience leads me to the conclusion that our elementary teaching is becoming more and more adapted to the necessity of pushing the very stupid boy through some examination. Compare the modern text-books of arithmetic and algebra with those in use thirty years ago, and you will find with what wealth of detail certain classes of sums are presented which are never found in real life outside an examination paper. These enormous collections of dull examples cannot fail to sicken the average intelligent boy, who never reaches anything interesting in mathematics unless he has a special aptitude for figures. Our net should be spread for the clever boy who is not a mathematician. The stupid boys will have to come in too, and the demigod needs no catching.

I think your syllabus excellent, supposing it were possible to start our whole system afresh with fully qualified teachers. This is not the case, and I feel that some of your proposed innovations might lead to disaster in the hands of a teacher not fully gifted with judgment. The use of logarithms and the slide-rule at an early stage of arithmetic seems dangerous, and I do not see why Napierian logarithms claim a place apart from their historical interest. (I suppose you mean true Napierian logarithms, and not those to the base *e*.) The rest of the course seems to be admirable: but I should feel rather bewildered if I were set down to teach it with the present text-books. We cannot change everything at once, and I feel convinced that the desired reform might be effected without departing far from the old order. Would it not be possible to take the course followed by modern text-books, which contain many good things, and strike out the unnecessary parts, at the same time inserting practical applications of the others? For instance, I fancy that elementary teachers would be quite ready to follow instructions such as these:—

*Arithmetic.*—Omit practice, and all weights and measures except avoirdupois, square and cubic measure, and the metric system.

Omit recurring decimals, which should be deferred until geometrical progression in algebra.

Use algebraic formulæ wherever convenient, being careful that the student understands the language of algebra.

Decimals to be taught with practical applications in the measurement of length, breadth, and area of pieces of paper. (Printed copies of examination papers serve admirably, and give many opportunities for teaching approximation: *e.g.* Find the number of square centimetres in a square inch correct to two places.)

The unitary method should not be allowed to obliterate all notions of ratio and proportion.

Area and volume to be taught, with practical applications in the measurement of given blocks of wood, &c., decimals being used freely, and a rule graduated both in tenths of an inch and in millimetres being employed. Mensuration and elementary algebra fall into their places here in conjunction with arithmetic.

*Algebra.*—The substance of the second book of Euclid to be taught with elementary formulæ. Squared paper, blocks, and cylinders to be used from the beginning.

Simultaneous equations to be illustrated with squared paper. More time to be given to problems and less to complex fractions. All artificial sums, common in examinations of the type of Oxford Responsions, to be cut out.

*Geometry.*—A course of geometrical drawings to be agreed upon which should replace Euclid II., IV., and VI. Books I. and III. to be taught still, but in conjunction with geometrical drawing. Euclid II. 12, 13, to be transferred to trigonometry, which should begin at an earlier stage.

*Mechanics.*—To be taught in conjunction with the other subjects, and used to illustrate them. A course of elementary physics, such as you give under the head of mensuration, could easily be made to fit into the mathematical course, so as ultimately to become an integral part of it.

I have put down these suggestions to indicate the kind of change that I think could be made with the least possible loss of what is good in the old system. One important point will be the reformation of examinations. The examiner will be always with us, and schools are bound to follow where he leads.

I know that my syllabus is cribbed from yours; but I think it follows the present lines of teaching more closely, and that it is desirable to do so as far as possible.

A. J. PRESSLAND: The conclusions which you form in your Glasgow address agree in the main with those at which I had arrived after a prolonged study of Swiss education.

Before I read your address, I had thus summarised the principal objects of mathematical reform:—

1. A careful selection of the material taught.
2. Uniform methods in elementary work.
3. A more rational treatment of the subject.

After fifteen years' experience in teaching, I am convinced that we cannot be too thorough in our reforms. I am equally convinced that most of a pupil's difficulties are due either to the language of the text-book or to the perverted ingenuity of the examiner. Boys of mathematical ability seldom see difficulties

unless these are pointed out to them by the master. Many boys of moderate attainments make but little advance, though they might reach to considerable proficiency if their powers of observation and their love of experiment were given sufficient scope.

The general lines on which these reforms should proceed are not hard to trace. The time-honoured separation of Arithmetic from Algebra must be ignored. More attention must be given to Mensuration and Computation. Euclid must be discarded. In drafting a Code Geometry to replace the latter, it might be assumed that some pupils have already had manual training and that all will take a course of Geometrical Drawing.

The other subjects which are taught at school might also be pruned. More especially is this the case in Dynamics, where verbosity has often replaced concrete illustration. But these subjects are scarcely under the undivided control of the schoolmaster, who must perforce seek guidance elsewhere.

By the introduction of these reforms much time will be gained. This should not be devoted to further specialised study in Mathematics, but to the acquirement of a good general education. The mathematical standard reached by the best of our schoolboys is high enough, that reached by the average boy is far too low. The former fails from lack of imagination and breadth of view, the latter from want of equipment. No greater mistake can be made than to believe that early specialisation is a cure for our defects.

At the Zürich Polytechnic, which is practically an international University for German-speaking students, I am told that the student from German State schools has an advantage over his fellows. This is ascribed not to a higher standard of attainments but to his more humanistic training. In the want of such a training may be found the origin of many of our mathematical oddities and conventionalities. Our present duty is not only to abolish them but to prevent their recurrence. To attain this end we must neither despise the criticism of the practical man nor hearken to the lamentations of the pedant.

Sir Philip Magnus: Professor Perry starts with the proposition that "It is usefulness which must determine what subjects ought to be taught to children and in what way," and he proceeds

to show how mathematics, which is a useful subject, should be taught with due regard to its "usefulness." This opens up a question of very grave educational significance. The word "useful" applied to a subject of education is generally employed somewhat loosely. A subject may be useful as applicable to some special purpose or need of life, or it may be useful as affording valuable mental discipline. To the engineer or architect mathematics is useful in the same way as is a knowledge of French or German to the commercial agent. But the question which Professor Perry asks us to consider is how may mathematics be usefully taught as a subject of general school education. It is important to bear in mind the difference in the meaning of the term "useful." A knowledge of physical science for the ordinary purposes of life is far more useful than any knowledge of mathematics beyond what is needed for keeping one's accounts. Yet it is generally recognised that science is not taught for the sake of the useful information the study affords, nor by the methods that accelerate our knowledge of scientific facts. Indeed, the usefulness of the results has tended rather to obscure the fact that the real value of science teaching consists in its usefulness as an instrument of training. It is necessary, therefore, that we should be quite clear as to our aims in teaching mathematics, in order that we may know how to teach it. If, as Professor Perry urges, mathematics is to be regarded as a tool to be used for obtaining certain practical results, we should, I think, at once accept most of his suggestions. But if the aim of mathematical teaching is rather brain development, accuracy in calculation and in measurement, the acquisition of correct methods of reasoning, or even a training that helps discovery, then I am not sure that Professor Perry's "royal road" is altogether a safe one for ordinary school children. It certainly seems to me that for boys who may not be intended on leaving school to be pitchforked into workshops, practice in the use of logarithms, without any knowledge of their relation to ordinary numbers, or dexterity in the use of the magic symbols $\frac{dy}{dx}$ and $\int$, without knowing what they mean, is of no great disciplinary value. Possibly, Professor Perry was not thinking of the ordinary boy or girl, who may never aspire to become an engineer, when he says: "If a man knows how to use

a weapon that is enough." When science lessons are given as a discipline, or when the use of tools is taught as manual training, one ought certainly to learn something more about one's weapons than the mere use of them, although one may not care to have the skill to forge them. So for the purposes of general education, a boy cannot be said to be learning mathematics by merely acquiring dexterity in the use of mathematical tables. I do not think Professor Perry quite meant all he said, but there is no doubt that he has been influenced in his suggestions by his long and successful experience in preparing students for technical pursuits

But although Professor Perry's methods are primarily of value in teaching mathematics to those who have to put their knowledge to some practical use, there is much in what he has told us of general applicability. When he says "that the proper method of teaching any subject is through some kind of experimental work," and that "it is not so much what the pupil is told, but what he discovers by himself that is of real value to him," he is enunciating pedagogic truths which have been too much lost sight of in the teaching of mathematics. It must not be forgotten, however, that the process of teaching a child to *discover* is not a rapid method of arriving at results, and may tend to retard rather than to quicken progress, as measured by the worked chapters of our school text-books. By introducing practical methods, such as the use of concrete geometrical figures and squared paper, and experimental work, the pupil's progress, as generally tested, would not be hastened, although his grip of mathematical science would undoubtedly be strengthened. Again, whilst agreeing with Professor Perry in the desirability of introducing, as soon as possible, the ideas underlying the differential and integral calculus, in so far, for instance, as they are illustrated by the direction of a curve at a point, and the calculation of areas, I do not see the use of taking a boy by forced marches through the ordinary exercises in differentiation and integration. Indeed, the aim of teaching should be rather to strengthen his faculties, and to supply a method of reasoning applicable to other subjects, than to furnish him with an instrument for solving practical problems.

With what Professor Perry has said about the use of Euclid as a text-book I am in complete accord. In a pamphlet on "The

Teaching of Geometry," published more than twenty years ago, I quoted from Professor Sylvester's address at a meeting of the British Association in 1869. "I should rejoice," he said, "to see Euclid honourably shelved, or buried 'deeper than did ever plummet sound' out of the schoolboy's reach;" and it is certain that had he lived till now he would have been in full sympathy with Professor Perry, for he told the members of the Mathematica Section over which he presided, "the early study of Euclid made me a hater of geometry." It is difficult to understand the permanence of Euclid's Elements as a school text-book. It is bad geometry and imperfect logic. Throughout Europe it has been abandoned, and nothing tends more to impede the proper study of Geometry in all our schools than the use of Euclid. Nevertheless, it remains. As a discipline, or training in logic, Euclid is defective; for not only is the reasoning faulty, but it is not the kind of reasoning used in sifting evidence and drawing conclusions in the affairs of ordinary life. When we are asked, "What book shall we substitute for Euclid?" the answer is, "None." Geometry must be taught like any other branch of science without the necessary reference to any particular text-book, and it must be taught practically. It is to be hoped that Professor Perry's renewed protest against the use of Euclid may be heard in our schools and class-rooms, and above all in our examination halls.

It seems to me that reform is almost equally needed in the teaching of Algebra. And here a large amount of "skipping might be allowed, but it should be the skipping of most of the exercises appended to the chapters of our text-books. These exercises serve only to secure dexterity in the manipulation of complicated expressions, which the student is very unlikely to require, and retard his advance to more interesting, and indeed more useful, parts of mathematics. There are few subjects which afford less mental discipline than Algebra as generally taught. In this respect it is far less useful than Arithmetic, which may be made a valuable means of training. Exercise in algebraic operations may afford dexterity in the use of Algebra as an instrument or tool, but such exercise is generally purely mechanical, and affords no stimulus to fresh thinking, which gives to any subject of study its real value in education.

As regards the teaching of Arithmetic, there is little to be said

which may not be found in what I fear is now almost a forgotten book—De Morgan's *Arithmetic.* To what De Morgan taught, no subsequent teacher has added much. For the teaching of Arithmetic, there should be constant appeals to mental results, and the pupil should be exercised in the endeavour to arrive at approximate answers before committing himself to rows of figures and complex operations. The early use of decimals should be encouraged, and the pupil should be shown at a very young age the similar relation of decimal fractions and decimal multiples to the unit. For instance, most beginners find a difficulty in seeing why $3 \times 3 = 9$ and $\cdot 3 \times \cdot 3 = \cdot 09$. No such difficulty would be experienced if the pupil were shown that 3 (units) $\times$ 3 (units) = 9 (units), and $3\bar{0} \times 3\bar{0}$ (where $\bar{0}$ denotes the unit) = $90\bar{0}$ in the same way as $\bar{0}3 \times \bar{0}3 = \bar{0}09$.

The teaching of mathematics in many of our schools has got into a groove, and has become a matter of mechanical routine. It neither exercises the mind of the teacher nor of the pupil. There is no subject in which reform of method is more needed, and Professor Perry has rendered a real service to education in directing thought to the subject.

Mrs. Mary Everest Boole: It is a common mistake to suppose that no preparation for science is needful or possible, except early teaching of what are called scientific subjects. Early attitude is far more important than early teaching.

Consciousness resides in but a small portion of the total machinery by means of which we think and learn, and it is dangerous and futile to constantly over-feed and over-exercise that small portion, while neglecting that larger portion whose action does not immediately cause consciousness. And, indeed, there is much reason to believe that the amount learned by children might be very much increased without the least injury to their health of body or mind, and with much less exertion than is now imposed on them and on their teachers, if the cultivation of the unconscious and that of the conscious portions of the organism were kept properly balanced and adjusted to each other. It is curious and painful to observe how many things have been proposed by true educationalists, simply for the purpose of ministering to the action of the unconscious mind, and afterwards

perverted, by persons possessed with the teaching mania, to the purpose of stuffing into children's minds some idea which is in the teacher's mind.

I fear we are in some danger of forgetting, in the rush of modern education, that conscious mental effort rather interferes with the work of establishing relations. Many persons seem to think that all the time is wasted for their children which is not spent in taking in consciously some special idea which some adult already understands. We must get rid of this notion entirely. There is a need of leisure time to renew our force for future work by getting our relations with nature, with man, and even with tools, more true, more perfectly harmonious, more elastic and easy, than is possible while the conscious brain is acting on the relation. Begin, therefore, as early as you can to set up in the child's mind what one may call a rhythmical alternation in science; a clear distinction between the time when he is being taught by man and the time when he is free to investigate or experiment as he pleases. Give him limits of time and place, lay down certain necessary negative conditions for safety and health, and to avoid annoying other people; and let the child realise that during that time, in the allotted place, provided he conforms to the prescribed conditions, no one will interfere with his experimenting exactly as he pleases.

The great dust raised over questions of detail about the best modes of teaching and the best books to use is blinding us all to the real main question at issue, which is: Are our children to learn science as people used to learn classics, by permission of a privileged caste of men and books who have a monopoly and know all about it? or are they to learn it because they are children of Nature, heirs of her kingdom and at home in her house, and because they therefore have a right to use her tools and her toys, her methods and her forces, subject only to such restrictions as she herself has laid on them? Let us settle that question first, and then details about what books to use, and how to teach this or that, will settle into their proper places.

MISS CHARLOTTE A. SCOTT: The request that I should state briefly any reflections to which Professor Perry's British Association address and Syllabus of mathematical studies give rise an

only be complied with, though I fear my comments will have very little value. As to the present extent of elementary mathematical teaching, and the details of the system for which this syllabus is suggested as a substitute, I have very little knowledge, as I have a very limited acquaintance with the routine of school work. Even my personal experience is of no service, as it relates only to the private boarding-schools for girls of thirty years ago. As to present-day teaching, I am familiar only with a certain class of results of school work, more particularly on this side of the Atlantic, where my work has lain for more than sixteen years. Here a teacher's work—or rather the work of any one school—is conducted with more freedom, and without the control of any recognised outside examination; though not without the examination influence, inasmuch as every school of any standing prepares its pupils for the entrance examinations of such colleges or universities as it finds convenient. I must confess that I cannot regard the results as encouraging to the advocates of leaving more freedom to the teacher. At my first introduction to American educational results, I was inclined to look upon the appreciation of geometry as in a higher ratio to the actual knowledge than in England; but it now seems to me that this is due rather to the smaller amount of actual knowledge than to the keener appreciation. There is a painful lack of facility and correctness in mechanical arithmetical and algebraic operations, even the simplest every-day ones; and if this facility is not acquired in the school course, when is it to be acquired? The work must be characterised in general as slipshod, a fact to which I believe the college teachers are fully alive, though the hindrance caused in college work is to a certain extent minimised by the natural quickness of the young American.

The general content of Professor Perry's syllabus seems well adapted to give an elementary working knowledge of mathematics; though I fear that teaching young students to rely so largely on immediate and apparent results obtained by graphical methods is not encouraging a very helpful attitude towards the larger problems of existence. The general idea of providing pupils as quickly as possible with a certain mathematical stock-in-trade may work some improvement in the nature of mathematical teaching, but I must say that I feel very dubious as to any good results

from any syllabus or system whatever. I should expect it to share the fate of "l'analyse des anciens et l'algèbre des modernes," of which Descartes says that "la première est toujours si astreinte à la considération des figures, qu'elle ne peut exercer l'entendement sans fatiguer beaucoup l'imagination; et on s'est tellement assujetti en la dernière à certaines règles et à certains chiffres, qu'on en a fait un art confus et obscure qui embarrasse l'esprit, au lieu d'une science qui le cultive." The natural tendency of a teacher seems to be to elaborate a system, and then administer it in regular doses. Treated in this manner, any system is deadly; and even the most inspiring syllabus will in time become a cut-and-dried system, unless some way can be devised for keeping elementary teachers in touch with enthusiastic investigators.

Professor D. E. Smith: As I digest the address, the following seems to be the essence:—

1. Mathematics are poorly taught in the schools of England.

As one who has seen more or less of such work in England, France, Germany, and America, I should say that the subject is more poorly taught in England than in any of the countries named, and that it is taught best in Germany.

2. It is absurd to say that the study is *useless*.

That depends on the definition of *useless*. Professor Perry defines *useful* to cover what others call the culture value, and is therefore safe, even in university mathematics.

3. Mathematics produce higher emotions and give mental pleasure, this phase being generally neglected.

This is true and is one of his strongest points. Unless the pupil loves the subject as he loves literature or art or chemistry, his teacher has lost a great opportunity.

4. The aid rendered by mathematics in studying the physical sciences has not been considered by teachers.

This is very true in England. The German schools do not neglect it so much. In the *Higher Arithmetic* (Boston, 1897), published by Professor Beman and myself, much attention has been given to this side of the subject but it is exceptional in America.

5. Mathematics have been taught almost solely for examination purposes.

This is a sad truth. England seems chained by her examination system. In France it is nearly as bad, although the examinations there seem more modern. In the eastern part of the United States (notably in New York State) we have followed the English system to our sorrow. In the Western States less attention is paid to examinations, and mathematics are better taught. Now that China has just reformed the examination system that has been an incubus for centuries, England and our Eastern States might humbly follow even in her footsteps.

6. Pupils are not taught to think in mathematics.

This is a corollary to the examination system. Teachers (unwisely) still feel that rules of thumb are the best equipment for examinations.

7. Problems should be modern, and appeal to the life of the pupil, in school or without.

This is axiomatic. But Professor Perry must convert his examiners. Look at any examination paper, and see the "applied problems."

8. England should adopt the metric system.

So should America. Both countries see Germany taking any amount of trade because she sells in international units. American manufacturers are likely soon to recognise this fact, and manufacture accordingly. They are beginning to taste the sweets of foreign trade, and this change will not be long in coming. How long before "the people" will use it is another question. Doubters, however, would do well to read Bigourdan's work on the metric system, published this year in Paris.

9. England should abandon Euclid.

Well, why not? She is about the only country left that uses it, and as one looks at the mathematics of the world since Newton's time he certainly cannot feel that the results of England's use of Euclid have been such as to render the monopoly necessary.

Indeed, as Professor Perry has defined "useful," there seems to me little to criticise in his paper. It is destructive, but he

remedies this defect by constructing a course. To be sure his course leans too much towards science, and away from common life, but that is easily remedied. The scheme is certainly better than the present one. But the obtaining of books, the training of teachers, the conversion of examiners, these are serious matters, and cannot be accomplished in one generation.

There is a tone running through the address that grates on one who loves mathematics for its own sake; but tones are intangible, and it is difficult to tell where this one has origin. It is a tone that seems to deride the love of mathematics *per se.* Professor Perry would make the theory of groups "useful," and quadratic residues, and Abelian functions, and anharmonic ratios, and invariants all "useful." To be sure the definition of "useful" helps him out, and his course is limited to secondary schools; but it is much like saying, "Read *In Memoriam*, and *The Ring and the Book*, and *The Iliad*, and *Piers Plowman*, because these are 'useful.'" But, after all, this may be a mere quibble over terms.

Certain it is that the speaker told a great deal of truth, and ii he went to an extreme, it was because in fighting conservatism one must sometimes be ultra-radical.

PROFESSOR HORACE LAMB: My difficulty in replying to you (apart from the pressure of other work) is that the address covers so much ground that it is hard to say anything pertinent to it as a whole. I quite agree that a sort of Kindergarten introduction to Geometry and Algebra is useful; but from the Secondary School stage mathematics should be more or less deductive in form. If it isn't, it ceases to be mathematics, and nothing is learnt from it (in a general way) that might not be better learnt from other subjects.

A good text-book to replace Euclid is much wanted; but it should be issued with some authority, and Cambridge is practically the only place that could confer this. It should also be revised every five or ten years, so that changes might be introduced gradually. Also it should be Euclidean in method, if not in phraseology. If anything else is attempted, a disastrous muddle will ensue, to be followed after a few years by a reversion to Euclid, as happened once, I think, in Italy. I have little to say

on the subject of the syllabus, except that, as a rule, I object to all programmes, and schedules, and "courses." A's "system" of teaching chemistry, B's "system" of teaching drawing, or C's of teaching the piano, may be very good as a method elaborated by A, or B, or C for doing his own work; but in other hands it is likely to become as mechanical and wooden as any other.

Professor G. B. Mathews: Such opinions as I have on the question of reform in the teaching of mathematics are based mainly upon twelve years' experience as a teacher of pure and applied mathematics in a provincial College. I have also had opportunities, as a mathematical examiner, of judging the quality of the work done by secondary and public schools both in England and Wales; and I have often discussed the teaching of mathematics with men actually engaged in it, from masters in elementary schools to University lecturers.

I shall confine myself to points which appear to me to be of primary importance, because questions of detail will be best worked out by schoolmasters themselves. If they are willing to make experiments, they will be able to find out better than any one else how far the reforms suggested by mathematical and physical experts are capable of being carried out in practice. Moreover, a schoolmaster has continually to bear in mind that mathematics is only a part of a school course, and must be considered in relation to the whole; a fact which the reformer, when a specialist, is occasionally inclined to forget. I am convinced that a truly liberal all-round education is in every way the best; and that disproportionate attention, at school, to any one subject, whether it be classics, mathematics, modern languages, or science, is very undesirable.

In criticising the results of school-teaching, as they have come under my notice, I shall try to point out what seem to me to be the principal *remediable* defects. It must be remembered that a certain proportion of students (perhaps a larger one than is generally supposed) cannot appreciate abstract mathematical reasoning at all; and their natural incapacity should not be laid to the charge of their teachers. But after making this allowance, I cannot but feel that many of the difficulties and disappoint-

ments I had with students attending my classes were largely due to the effects of injudicious training at school.

The worst and most disappointing students were those who had been conscientiously taught upon what I will call the imitation system. They had learnt, say, with servile fidelity to their text-book, two or even three books of Euclid, and perhaps a small number of deductions, got up in the same way; they had acquired the knack of doing "sums" (as they invariably called them) in arithmetic and algebra by fixed rules and in imitation of typical examples. As a rule, they had lost all power of initiative: they could not even try to think; they could take admirable notes of a lecture, but could not profit by them in the least; and whatever interest in the subject they originally may have had was either perverted or extinct. A large proportion of this pathetically hopeless class were normal students or English girls—the latter sometimes coming from schools of considerable reputation. In a few exceptional cases the student's natural capacity was so great as to survive in spite of all the blighting influences to which it had been exposed, and the effect of a change of method was then very remarkable. An intelligent girl, who ultimately did herself and our College great credit as a science student, declared that for nearly the whole of her first session, and in all her subjects, she was mainly occupied in changing her attitude of mind, and methods of study, so as to profit by the really effective teaching which she felt was at last within her reach.

Slovenly and inaccurate habits of thought and work produced trouble of another kind. Both the source and remedy of this are obvious, and I only mention it because habits of this kind, once inveterate, are difficult, and sometimes impossible, to cure.

Students, even adults (and I had them of all ages ranging from sixteen to over thirty), very rarely came with any idea of mathematics as a science; they had no sense of perspective, and had never been taught the relative importance of what they had learnt, except possibly from the standpoint of examination requirements. As some of my best pupils came in the condition of raw beginners, or even worse (the little they professed to know being all wrong), and as I found whole classes easy to interest in general principles, such as symmetry, and the relations between different subjects, such as arithmetic and algebra, or geometry

and analysis, I believe that some improvement in this direction can be effected even in schools. Of course, a boy, and especially a young boy, cannot be expected to grasp broad principles in the same way as an older student; but things can be put before him in such a way as to suggest connections and analogies, to impress him with what is really fundamental, and protect him from misapprehensions and wrong points of view.

To almost all my students it was a revelation to learn that mathematics is a living, progressive, and experimental science like physics and chemistry; and that, like them, it has important practical applications. Their only idea of practical mathematics, as a rule, was arithmetic of the basest commercial kind. They knew "rule of three," but could not apply it to chemical calculations; they were ignorant of the commonest formulæ of mensuration; they had never seen a book of mathematical tables; did not know the metric system; and failed utterly in the approximate calculations required in the physical laboratory.

In saying this I am only echoing the complaint which is being made by Professor Perry and other eminent teachers of science, and, with a certain reservation, I heartily endorse their criticism. Far too much school time is spent upon exercises which, from *every* point of view, are thoroughly artificial and unpractical; and the fact that such useless trumpery should still pervade our text-books is most deplorable. With a few honourable exceptions, it is hardly possible to open a school text-book on arithmetic or algebra without finding examples of a kind which never have occurred and never will occur naturally in any theoretical investigation or practical application. This is a crying evil, and urgently requires amendment. The direction in which reform is possible and desirable has been, I think, sufficiently indicated, and I do not go into details here.

But I should like to state the reservation which affects my sympathy with a certain section of the reformers. There is an occasional tendency to exalt the practical man at the expense of a more or less imaginary being called "the mere mathematician." Mathematicians, as such, are so accustomed to their ignoble status in this country, that a gibe, even at the hands of those whom they most benefit, is not likely to make them wince. But, while the personal question is negligible, there is an important

question of principle involved. If a physicist or engineer is to make independent and intelligent applications of mathematics, he must have some real knowledge of mathematical principles, and a mere acquaintance with rules and formulæ is not sufficient. For several years I gave a course of elementary calculus to a class of electrical engineering students, who were learning the theory and practice of dynamos in the physical department. Naturally, I chose, as well as I could, appropriate illustrations and exercises; but I tried to teach the *principles* of the subject, just as I should have done with any other class. And I am convinced that the theoretical knowledge which the students obtained, though not extensive, was of the greatest benefit to them; and I am glad to know that some of them, at any rate, continued to study the calculus after becoming engaged in practical work.

A great deal has been said of late years about text-books and syllabuses. No doubt these are important matters: but the main point is to improve the character of the teaching. This is specially true of schools, where the text-book, at any rate, is mainly used for the exercises contained in it. Provided that the exercises are suitable, and that the book does not contain matter likely to mislead the exceptional boy who may happen to read it, a competent teacher will be content. The main benefit to be expected from improvement in text-books is the awakening of teachers to new ideas. In the same way a well-considered syllabus is a useful guide to teachers, if they do not make a fetish of it; as for the pupil, he should be kept, as far as possible, from seeing or thinking of a syllabus. And it must be remembered that the adoption of the best of syllabuses does not result, of itself, in better teaching. I feel sure that if Professor Perry's syllabus, admirable as it is from his point of view, and unobjectionable from any other, were universally adopted to-morrow, the unintelligent mechanical teacher would go on exactly as before. For one kind of badness another would be substituted, and that would be all.

Fortunately, the supply of competent teachers appears to be increasing, and many of them have shown themselves willing, even anxious, to consider practical suggestions of reform. The main obstacle now, I think, is, not the attitude of schoolmasters, but that of the examining boards which so largely control the

secondary education of England. I do not propose to abolish examinations: I believe that the Oxford and Cambridge Local Examinations, and those of the College of Preceptors, have served a very useful purpose, and may still continue to do so, if they keep in touch with the progress of ideas, and are not petrified by the subtle agency of routine. It is hopeless, apparently, to try to upset the prevalent notion that the best, if not the only, test of a teacher's efficiency is by the verdict of an external examiner. So long as this is the case, examinations will control teaching, and not *vice versâ*. This being so, it is surely reasonable to ask that examinations should not put a premium upon inferior methods of teaching, and encourage waste of time upon futilities, while other things, no harder to teach, and infinitely more valuable and interesting, are persistently ignored. To take one single illustration out of many that could be given: an arithmetic paper for boys over fourteen is almost certain to contain an example or two on recurring decimals, and problems which work out "neatly" by the use of vulgar fractions. These have to be anticipated by the teachers: the result is that boys learn to hate decimals, and fail entirely to see the proper use of them: they talk of the "principle of cancelling"; and they have a rooted conviction that no answer can be right which does not "come out,"—that is to say, present itself in an attractively simple form.

The best hope I can see for a real and immediate improvement depends upon the action of influential schools. Suppose several schools of established reputation give a fair trial of rational methods and a reformed programme with the bulk of their boys from the lowest forms upwards; let them try the effect for themselves, and if they must have an external examiner, let them procure one (as they can easily do, if they wish) who is not actually prejudiced in favour of routine. If, after allowing sufficient time (say three years) for the experiment, they find, as I am sure they will, that the results are eminently satisfactory, they will give the reformers a powerful argument with which to approach the examining boards. For if these conservative bodies object, as they very likely will, that the reformed programme is Utopian, and the work of irresponsible professors who know nothing of the conditions of school teaching, the answer will be ready that the verdict of those who are best qualified to judge is in favour of

a change, because they have tried both systems under the same conditions, and have found the new one better than the old.

I am glad to know that Winchester College has been recently moving in this direction: the current programme of mathematical study in the school is still (and I hope may always be) liable to modification, but in its present form indicates a great advance, and ought to satisfy all but extreme partisans. It is perhaps as good as can be expected in the present condition of affairs; and there is no doubt, I think, that, so far as graphical methods, geometrical drawing, and other such innovations, have been introduced, the results have been decidedly encouraging.

## PROFESSOR PERRY: REPLY.

Several speakers, ignoring what I have written and sent them about England's neglect of science and true education, say that I think of every boy as if he were going to be an engineer or manufacturer. This is partly due to my assumption that certain facts were notorious. I will here state one of these almost in the words of an experienced Oxford man, for it may be that mere ignorance of such facts may delay reform. For the pass degree at Oxford, for the academic distinction which enables men to be engaged as schoolmasters who have to teach geometry and other subjects, there is a test: to have off by heart the propositions of two books of Euclid. This is an interesting result of the long-continued attempt to teach mathematics on what are called orthodox lines. To be able to repeat any proposition of Euclid, knowing that if in Simpson's Euclid the letter B is used for a particular point on a particular figure, it is not allowable to use the letter P or X instead! To know that one will not be asked any question whatsoever about the nature of the proof that one has got off by heart! To know that one cannot be asked a *rider* on any proposition!

Oxford has, then, frankly admitted the worthlessness of the orthodox methods of mathematical study in education; and when she is laughed at for compelling candidates to repeat gibberish which does not even rhyme, it ought to be remembered that there is a method underlying her madness. I consider that of

many evils she has chosen the least. She loses in reputation among the general, but she has never cared for reputation outside herself; she hurts the memory of every candidate, perhaps permanently, but her system leaves his mind and soul untainted. The average Englishman hates school education, because every specimen of it that he has seen offends his common-sense. And Oxford hates all ideas of education through mathematics and natural science, because what she has seen in that way, everywhere, has been offensive to her common-sense. I feel that Oxford may be the first to take to education through mathematics and natural science when she finds that such an education is really possible for average boys.

Principal Rücker says that the English people *will have* examinations. This is so, but what kind of examinations? The English people bore for centuries with a great abuse, the entrance of men to professions and the civil and military services by the rules of trades unions and by patronage; and when at length the abuse was too awful to be borne with longer, they declared that there ought to be tests of fitness applied. So the wise men of the country have applied the best tests they could think of. It is surely our duty to educate the wise men so that these tests of fitness shall be real, shall induce real education of every kind in candidates. The word "examination" has become a technical term for a written examination by an outside examiner, who fixes the curriculum irrespective of the inclinations of pupils and teacher, who sets questions without the assistance of the teacher, who makes a report of fitness of pupils and teacher without inspecting the school, without seeing the pupils at work, without even having any personal acquaintance with the teacher. I do not think that this is what the English people asked for or want (except high school mistresses who would wail if we took away from them their golden calf, their external standard). Several speakers ask for a reform of examinations, but I assert that we want reform of a whole system of dunce manufacture of which examination s only one part.

Is the reform to come from above or below? In Germany, reforms [1] always come from above. In England, they come

[1] Yes, and the reverse of reform! The "State" in Germany can do what it pleases. The "Prussian Regulations" of 1854 almost destroyed elementary education; the "General Provisions" of 1872 rehabilitated it.

from below, and from above, and from the middle. We must educate everybody. The whole system of teaching and the testing of teaching is absurd, and we must educate parents and teachers and examiners and the wise men who appoint examiners.

The fact is, this business of school education is a new one for the English race. The higher classes have had such book learning as they wanted; they and the middle classes learnt to read and write and cipher a little, but there was no education in all this. What the English people want they can always get, and they never did want this. Real education in England was given through sport, civil and foreign wars and preparation for them, adventures by land and sea, acquaintance with Nature, and by mental and emotional struggles about religion and liberty. English boys received their education not through pot-hooks and hangers, the Latin Grammar and Euclid—if there had been education in these, England would have covered itself with schools from Land's End to Berwick, from Yarmouth to Milford. It is the most pathetic English humour that one finds in the scene between Mrs. Page and Sir Hugh Evans and the boy. The average Anglo-Saxon boy has never been educated except through observation and trial. If you want in this twentieth century to develop his brain and also to give him that kind of knowledge which he must have if he is to compete with men of other races, you will indulge him to the uttermost in that method of study which he loves—observation and experiment and discovering for himself. Every examination you hold is an offer to bribe him in one way or another to pretend to believe in all sorts of scholastic things. The test of Argus was that he should recognise his king in the garb of a beggar. The test of our wise men ought to be their power to recognise the great genius of our race when it reveals itself in what is called the stupidity of the average English boy; a power which has defied the tuppeny-ha'penny educational measures of our superior persons for more than a century: the power to stand being bullied in every generation without caving in. His race now recognises his ignorance of necessary scientific principles, and his want of scientific—that is systematised Anglo-Saxon—method, face to face as he is with hungry and envious competitors in the struggle for existence, but we may be sure that it will reject any remedies that are foreign to his nature. What is the use of insisting on his taking a German remedy when his constitution is

utterly different from that of the German? Slavery was extinguished in England five centuries ago: up to 1809 the German peasant was sold with the land that he tilled. A hundred and fifty years ago there was no German literature: English literature is as great as that of any country that has ever been. This makes a great difference in the educational value of ancient classics. As an admirer of Germany, I do not like to refer to even one of the many things of this kind that ought to be taken into account. They go to explain why we were never the slaves to an idea, although we indulge in many prejudices; why we are obedient to no one thinker; why with us reform comes from no one quarter; why it is that in every village of England one or two cranks, men of individuality, are to be found, men who think and try things for themselves and who become leaders in times of emergency; why it is that we are slow and sure, and why it is that our whole history has been one of persistent observation and trial. We are very cautious in adopting things from other nations.[1] We let things slide. Our minds lie fallow for a generation, and we help outsiders to laugh at our laziness; but after our Sabbath we concentrate our thoughts on some great object, and carry it through to the world's astonishment and admiration. The examination system of our days is a home-made nostrum, whose effects we are observing with lazy, good-humoured forbearance. It costs a lot of time and money, and we mean to get from the experiment all that it can teach us, and we know that we shall come to right methods in the long run. Some of us think that the nation is too deliberate, and risks too much by making experiments on too large a scale. Another race really less capable may in these modern times get such a veneer of method imposed upon it by its rulers in even one century as to nearly ruin us in the struggle for existence, while we are calmly expecting the mushroom civilisation to decay with the same rapidity as that of its growth. If we are too slow and unscientific, surely t is because the wise men of the nation are not wise enough or earnest enough or enthusiastic enough to get their advice followed;

[1] Our caution in not adopting standing armies 300 to 400 years ago saved England's liberties; or, rather, gave us greater ease in retaining our liberties. Our caution in adopting German methods of dunce manufacture may prevent our losing something that no standing army could have taken from us. I think that Germany has lost it already, in spite of her success in turning out a certain kind of chemist.

they have not enough faith; they are impatient and easily disheartened. We laugh at people who write to the *Times* and get up discussions, but in truth the safety of England in all the ages has been due to the existence within her of so many of these cranks. I wrote to the *Times* about six months ago against the iniquity of the external examination system of London University being applied to the constituent colleges. Dr. Armstrong has written a better letter lately. Surely others also will write! If we carry this point in the University of London, it will lead to the greatest possible reform in the examination system of Great Britain. The Scotch Education Board has reformed in the very largest way its whole examination system in all primary and continuation, in all evening and all secondary schools except two; surely a very interesting experiment for English people to watch. If ever there was a case for external examination, it existed in the Intermediate Board for Ireland; but that system is not a success, and its reform is being considered. Of the Civil Service Commission I know only by hearsay. They are asked to adopt the best method of finding the fittest among candidates for the Civil Service; they can ask for greater freedom from Parliament if they need it; they can resign if their notions are not carried out. Principal Rücker suggests in the words "my people will have it so," that, however bad an examination system may be in the eyes of all the wise men of the country, in the eyes of every writer of every newspaper in the country—for the scorn of the system fills the land—that the English people will cling to it, that "my people will believe a lie."

Dr. Larmor approves of my notions of reform, but objects to the increased cost, fewer pupils for each teacher, and each teacher a more expensive man. Does, then, the British nation think that it is actually the wages of the teacher which costs all the money spent in schooling? In the days of dames' schools an old lady explained: "It's little they pays us, and it's little us teaches them;" and this curse of small wages is still upon us. He is a poor restaurant keeper or boarding-house keeper who does not get better wages than the master of a lower form in many schools, and yet the lower forms need the most intelligent teachers. Notice when such a master becomes a boarding-house keeper how his income is increased 500 or 800 per cent. Teaching such

a form is often as easy as turning a barrel-organ, and the master gets organ-grinder's wages. Keeping a boarding-house requires thought. Dr. Larmor's remedy is that examiners should get much higher wages. I agree with him, but I say that it is only one of the remedies. Every teacher in the country ought to get much higher wages: for the most important kind of work in the country there ought to be such wages as will attract men of common-sense.

We who have taken part in this discussion have been criticised by some educationists because we have only been expressing well-known educational truths. They forget that, however well-known these truths may be, they have never yet—never till now—been expressed publicly by more than two or three mathematical teachers. They forget that a reform in the teaching of mathematics was absolutely impossible without the consent and advice of the mathematicians.

It will be found that my syllabus contains almost all the new suggestions which were made by speakers who had no time to study it. (1) Experimental geometry to precede demonstrative. (2) Some deductive reasoning to accompany experimental geometry. (3) Mathematics to enter into the experimental science syllabus as much as possible. (4) Rough guessing at lengths, weights, &c., to be encouraged. (5) Recognition of the incompleteness of any external examination. (6) The importance of familiarising a boy with problems in three-dimensional space. (7) A hard and fast syllabus undesirable; even the sequence of subjects to be left to a good teacher's initiative. If my critics will consult the new syllabus in Practical Geometry, of the Board of Education, they will see that a very great reform has been effected in this subject, and that it is no longer a mere collection of rules: it is now really an educational subject allowing a student to let his mind and imagination develop in several of many directions, and bringing him into contact with many subjects through the common-sense application of a very few general principles.

Professor Forsyth and other critics have attacked what are misprints or verbal errors in my address, and they have also attacked ideas which are not in my address at all. I do not feel flattered when I find that they think I advocate giving *all* existing

knowledge to a boy; or that when I speak of a man's having command of a *tool*, I mean that he has only bought the tool or stolen it without understanding how it is to be used; or that when I speak of *utility*, I think only of a portion of one of the eight definitions of utility which I gave so carefully.

I answer many of the remarks of my critics when I say that I am in perfect accord with every opinion as to the teaching of mathematics expressed by Professor Forsyth. I wish I could think he agreed with everything said by me; but he certainly agrees with my address and my syllabus in every matter to which he referred. If Dr. Lodge has any plan better than mine of making engineers give up their use of mere formulæ or other tools which they do not understand, I should like to know it: his object is mine. I have tested the efficiency of my method over and over again. I write badly, but I would ask him to look for the spirit behind my poor language. Professor Hudson is perfectly right in thinking that I agree with every opinion which he has expressed.

In all that Mr. Langley says, he is in accord with me; I wish that such an experienced teacher had criticised more of my syllabus. Professor Everett goes very fully into my whole scheme, and is in perfect agreement with me on every point. Lord Kelvin, without mentioning details, is thorough in his general commendation of my address and my syllabus. Principal Rücker agrees with me in all that he says; so does Dr. Silvanus Thompson, who goes farther. I am in enthusiastic accord with every word of Mrs. Shaw and her husband. For many dismal, strenuous years, I have followed or assisted Oliver Heaviside, Professor Henrici, Professor Greenhill, and Professor Minchin in mathematical reform, and I knew that they would support me in essential things. I should feel more dismal now if I thought that they would subscribe to any fixed syllabus either of mine or of their own. We have only one formula: to give a student every chance of self-development. Professor A. Lodge, like his brother Principal Oliver Lodge, is, I know, in even better agreement with my whole scheme than their few words express. Professor Miall, as he is almost the foremost educationist of our time, is of course in accord with me, and so is Professor Jamieson. Mr. Cooper does not seem to see that he agrees with me in principle.

I cannot easily express my pleasure in knowing that Sir John Gorst sympathises so deeply with the proposed reform, although I know that he will not exercise his great power of making the reform unless it is approved of by other mathematicians. I am glad that Mr. Workman agrees with me in principle; he, like others, sees difficulty in carrying out my plans. I admit some difficulty, but rather in another direction—the difficulty inherent in any large reform. Dr. Sumpner is one of many principals of Technical Colleges or Polytechnics who have welcomed the reform, and who give me much encouragement by their reports as to the enthusiasm and good attendance of evening students who can hardly be induced to attend ordinary mathematics classes. Encouragement from a man in the position of Mr. Eggar is exceedingly valuable to me; he is right in assuming, with Professor Hudson and many others, that I ask only for agreement with my general principles, and that I would give much freedom to any teacher who wanted it. He sees that I have only one fear—fear of creating a crystallised system, an unchangeable ritual; fear of allowing teachers to think that they can place their own responsibility on the shoulders of anybody else. Mr. Eggar's suggestions ought to be of particular value to masters in public schools, as also ought those of Mr. Pressland. Mr. Pressland has made a very complete examination of the work done in foreign schools in preparation for Polytechnic training; I feel sorry that he is so perfectly in agreement with what I said on this subject. I wish that Mrs. Boole had written at greater length. All that she has published on education is of great value, and particularly her paper on "The Preparation of the Unconscious Mind for Science" (Kegan Paul, Trench, Trübner, and Co.). It is not the smallest service done me by this discussion that it has led me to acquaintance with her writings.

On the whole, I think it may be said that I am in accord with every one of my critics, but of course I know that they cannot unreservedly agree to the adoption of my syllabus as it stands for every kind of student. At all events, it is quite evident that there is unanimity in the desire for an immediate large reform in the teaching of mathematics.

I have long known that there is this unanimity among educationists generally, but it is unexpected to find it among the

great mathematicians, and the most important teachers of mathematics. I take it that we are all agreed upon the following points :—

1. Experimental methods in Mensuration and Geometry ought to precede demonstrative geometry, but even in the earliest stages some deductive reasoning ought to be introduced.

2. The experimental methods adopted may greatly be left to the judgment of the teacher: they may include all those mentioned in the Elementary Syllabus which I presented.

Some of the things for which I contend were put so prominently forward that, if speakers did not object to them specifically, they may almost be taken as being agreed to. They are such things as these that follow: most of them are agreed to specifically by about half my critics.

3. Decimals ought to be used in arithmetic from the beginning.

4. The numerical evaluation of complex mathematical expressions may be taken up almost as part of arithmetic or at the beginning of the study of algebra, as it is useful in familiarising boys with the meaning of mathematical symbols.

5. Logarithms may be used in numerical calculation as soon as a boy knows that $a^n \times a^m = a^{n+m}$, and long before he is able to calculate logarithms. But a boy ought to have a clear notion of what is meant by the logarithm of a number.

6. In mathematical teaching a thoughtful teacher may be encouraged to distinguish what is essential for education in the sequence which he employs, from what is merely according to arbitrary fashion, and to endeavour to find out what sequence is best, educationally, for the particular kind of boy whom he has to teach.

7. Examination cannot be done away with in England. Great thoughtfulness and experience are necessary qualifications for an external examiner. It ought to be understood that an examination of a good teacher's pupils by any other examiner than the teacher himself is an imperfect examination.

I have not much doubt as to the unanimity with which everybody may be said to have agreed explicitly or implicitly to all the above statements. About these that follow I am in more doubt. More than half my critics will, I believe, agree to them all for all students. I think that every one of my critics will agree to allow

a judicious teacher a free hand, especially when he knows that his pupils are likely to need the use of mathematics in their other studies, and especially if they are likely to become engineers—that is, men who apply the principles of natural science in their daily work.

8. A thoughtful teacher ought to know that by the use of squared paper and easy algebra, by illustrations from dynamics and laboratory experiments, it is possible to give to young boys the notions underlying the methods of the Infinitesimal Calculus.

9. A thoughtful teacher may freely use the ideas and symbolism of the Calculus in teaching elementary mechanics to students.

10. A thoughtful teacher may allow boys to begin the formal study of the Calculus before he has taken up advanced algebra or advanced trigonometry, or the formal study of analytical or geometrical conics, and ought to be encouraged to use in this study, not merely geometrical illustrations, but illustrations from mechanics and physics, and illustrations from any other quantitative study in which a boy may be engaged.

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*To the Committee appointed by the British Association to report upon the Teaching of Elementary Mathematics.*

GENTLEMEN,

At the invitation of one of your own body, we venture to address to you some remarks on the problems with which you are dealing, from the point of view of teachers in Public Schools.

As regards Geometry, we are of opinion that the most practical direction for reform is towards a wide extension of accurate drawing and measuring in the Geometry lesson. This work is found to be easy and to interest boys; while many teachers believe that it leads to a logical habit of mind more gently and naturally than does the sudden introduction of a rigid deductive system.

It is clear that room must be found for this work by some unloading elsewhere. It may be felt convenient to retain

Euclid; but perhaps the amount to be memorised might be curtailed by omitting all propositions except such as may serve for landmarks. We can well dispense with many propositions in the first book. The second book, or whatever part of it we may think essential, should be postponed till it is needed for III. 35. The third book is easy and interesting; but Euclid proves several propositions whose truth is obvious to all but the most stupid and the most intellectual. These propositions should be passed over. The fourth book is a collection of pleasant problems for geometrical drawing; and, in many cases, the proofs are tedious and uninstructive. No one teaches Book V. A serious question to be settled is—how are we to introduce proportion? Euclid's treatment is perhaps perfect. But it is clear that a simple arithmetical or algebraical explanation covers everything but the case of incommensurables. Now this case of incommensurables, though in truth the general case, is tacitly passed over in every other field of elementary work. Much of the theory of similar figures is clear to intuition. The subject provides a multitude of easy exercises in arithmetic and geometrical drawing; we run the risk of making it difficult of access by guarding the approaches with this formidable theory of proportion. We wish to suggest that Euclid's theory of proportion is properly part of higher mathematics, and that it shall not in future form part of a course of elementary geometry. To sum up our position with regard to the teaching of geometry, we are of opinion—

1.—That the subject should be made arithmetical and practical by the constant use of instruments for drawing and measuring.

2.—That a substantial course of such experimental work should precede any attack upon Euclid's text.

3.—That a considerable number of Euclid's propositions should be omitted; and in particular

4.—That the second book ought to be treated slightly, and postponed till III. 35 is reached.

5.—That Euclid's treatment of proportion is unsuitable for elementary work.

Arithmetic might well be simplified by the abolition of a good many rules which are given in text-books. Elaborate exercises in vulgar fractions are dull and of doubtful utility; the same amount

of time given to the use of decimals would be better spent. The contracted methods of multiplying and dividing with decimals are probably taught in most schools; when these rules are understood, there is little left to do but to apply them. Four-figure logarithms should be explained and used as soon as possible; a surprising amount of practice is needed before the pupil uses tables with confidence.

It is generally admitted that we have a duty to perform towards the metric system; this is best discharged by providing all boys with a centimetre scale, and giving them exercise in verifying geometrical propositions by measurement. Perhaps we may look forward to a time when an elementary mathematical course will include at least a term's work of such easy experiments in weighing and measuring as are now carried on in many schools under the name of Physics.

Probably it is right to teach square root as an arithmetical rule. It is unsatisfactory to deal with surds unless they can be evaluated, and the process of working out a square root to five places provides a telling introduction to a discourse on incommensurables; furthermore it is very convenient to be able to assume a knowledge of square root in teaching graphs. The same rule is needed in dealing with mean proportionals in Geometry.

Cube root is harder and should be postponed until it can be studied as a particular case of Horner's method of solving equations approximately.

Passing to Algebra, we find that a teacher's chief difficulty is the tendency of his pupils to use their symbols in a mechanical and unintelligent way. A boy may be able to solve equations with great readiness without having even a remote idea of the connection between the number he obtains and the equation he started from. And throughout his work he is inclined to regard Algebra as a very arbitrary affair, involving the application of a number of fanciful rules to the letters of the alphabet.

If this diagnosis is accepted, we shall be led naturally to certain conclusions. It will follow that elementary work in algebra should be made to a great extent arithmetical. The pupil should be brought back continually to numerical illustrations of his work. The evaluations of complicated expressions

in $a$, $b$, and $c$ may of course become wearisome; a better way of giving this very necessary practice is by the tracing of easy graphs. Such an exercise as plotting the graph $y = 2x - \frac{x^2}{4}$, provides a series of useful arithmetical examples, which have the advantage of being connected together in an interesting way. Subsequently, curve-tracing gives a valuable interpretation of the solutions of equations. Experience shows that this work is found to be easy and attractive.

With the desire of concentrating the attention of the pupil on the meaning rather than the form of his algebraical work, we shall be led to postpone certain branches of the subject to a somewhat later stage than is usual at present. Long division, the rule for H.C.F., literal equations, and the like, will be studied at a period when the meaning of algebra has been sufficiently inculcated by arithmetical work. Then, and not till then, will be the time to attend to questions of algebraic form.

But at no early stage can we afford to forget the danger of relapse into mechanical work. For this reason it is much to be wished that examining bodies would agree to lay less stress upon facility of manipulation in Algebra. Such facility can generally be attained by practice, but probably at the price of diminished interest, and injurious economy of thought. The educational value of the subject is sacrificed to the perfecting of an instrument which in most cases is not destined for use.

To come to particulars, we think that undue weight is often given to such subjects as algebraic fractions and factors. The only types of factors which crop up continually are those of $x^2 - a^2$, $x^2 \pm 2ax + a^2$, and, generally, the quadratic function of $x$ with numerical coefficients.

In most elementary Algebra books there is a chapter on Theory of Quadratic Equations in which a good deal of attention is paid to symmetric functions of roots of quadratics. No further use is to be made of this till the analytical theory of conics is being studied. Might not the theory of quadratics be deferred till it can be dealt with in connection with that of equations of higher degree?

Indices may be treated very slightly. The interpretation of negative and fractional indices must of course precede any

attempt to introduce logarithms; but when the extension of meaning is grasped, it is not necessary to spend much more time on the subject of indices; we may push on at once to the use of tables.

It will be seen that our recommendations under the head ot Algebra are corollaries of two or three simple guiding thoughts; the object in view being,—to discourage mechanical work; the means suggested,—to postpone the more abstract and formal topics and, broadly speaking, to arithmeticise the whole subject.

The omission of part of what is commonly taught will enable the pupil to study, concurrently with Euclid VI., a certain type ot diluted trigonometry which is found to be within the power of every sensible boy. He will be told what is the meaning of sine, cosine, and tangent of an acute angle, and will be set to calculate these functions for a few angles by drawing and measurement. He will then be shown where to find the functions tabulated, and his subsequent work for that term will consist largely in the use of instruments, tables, and common-sense. A considerable choice of problems is available at once. He may solve right-angled triangles, work sums on "heights and distances," plot the graphs of functions of angles, and make some progress in the general solution of triangles by dividing the triangle into right-angled triangles. Only two trigonometrical identities should be introduced—$\sin^2\theta + \cos^2\theta = 1$, and $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$. In short, the work should be arithmetic, and not algebra.

Formal Algebra cannot be postponed indefinitely; perhaps now will be the time to return to that neglected science. We might introduce here a revision course of Algebra, bringing in literal equations, irrational equations, and simultaneous quadratics illustrated by graphs, partial fractions, and binomial theorem for positive integral index. Side by side with this it ought to be possible to do some easy work in mechanics. Graphical statics may be made very simple; if it is taken up at this stage, it might be well to begin with an experimental verification of the parallelogram of forces, though some teachers prefer to follow the historical order and start from machines and parallel forces. Dynamics is rather more abstract; a first course ought probably to be confined to the dynamics of rectilinear motion.

It is not necessary to discuss any later developments. The plan we have advocated will have the advantage of bringing the pupil at a comparatively early stage within view of the elements of new subjects. Even if this is effected at the sacrifice of some deftness in handling *a*, *b*, and *c*, one may hope hat the gain in interest will be a motive power of sufficient strength to carry the student over the drudgery at a later stage. Some drudgery is inevitable, if he is ultimately to make any use of Mathematics. But it must be borne in mind that this will not be required of the great majority of boys at a Public School.

We beg to remain, gentlemen,

Yours faithfully,

G. M. BELL, WINCHESTER.
H. H. CHAMPION, UPPINGHAM.
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# BRITISH ASSOCIATION

FOR THE

# ADVANCEMENT OF SCIENCE

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BELFAST, 1902

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*Teaching of Elementary Mathematics.—Report of the Committee, consisting of* Professor CHRYSTAL, Mr. W. D. EGGAR, Mr. H. W. EVE, Professor FORSYTH (*Chairman*), Professor GIBSON, Dr. GLADSTONE, Professor GREENHILL, Professor R. A. GREGORY, Professor HENRICI, Professor HUDSON, Dr. LARMOR, Professor A. LODGE, Principal O. LODGE, Professor LOVE, Major MACMAHON, Professor MINCHIN, Professor PERRY (*Secretary*), Principal RÜCKER, Mr. ROBERT RUSSELL, *and* Professor S. P. THOMPSON, *appointed to report upon improvements that might be effected in the teaching of Mathematics, in the first instance in the teaching of Elementary Mathematics, and upon such means as they think likely to effect such improvements.* (*Drawn up by the Chairman.*)

There were seven meetings of the Committee.

APPENDIX.—*Two Suggested Schedules of Experimental Geometry.*

IN submitting their present report, the Committee desire to point out that this is not the first occasion on which the British Association has attempted to deal with the teaching of elementary mathematics. About thirty years ago, a similar body was appointed to consider a part of the subject, viz., "the possibility of improving the methods of instruction in elementary geometry"; and two reports were presented, one at the Bradford meeting in 1873 (see the Report volume for that year, p. 459), the other at the Glasgow meeting in 1876 (see the Report volume for that year, p. 8).

The two reports advert to some of the difficulties that obstruct improvements in the teaching of geometry. One of these is alleged to be "the necessity of one fixed and definite standard for examination purposes"; apparently, it was assumed that this fixed and definite standard should not merely be required from all candidates in any one examination, but also be applied to all examinations throughout the country. In order to secure the uniformity thus postulated, the Committee, thinking that no text-book had been produced fit to succeed Euclid in the position oı authority, and deeming it improbable that such a book could be produced by the joint action of selected individuals, suggested the publication of an authorised syllabus. In their second report, they discussed the merits of a particular syllabus—that of the Association for the Improvement of Geometrical Teaching; but, in spite of such commendation as was then expressed, the syllabus has not been generally adopted.

It is still true that (in the words of the former Committee) "in this country at present teaching is guided largely by the requirements of examinations." For some time to come, the practice of the country is not unlikely to allow examinations to retain at least a partial domination over teaching in schools. Accordingly, if the teaching is to be improved, it seems to be a preliminary requisite that examinations should be modified; and, where it is possible, these modifications in the examinations should leave greater freedom to the teacher, and give him more assistance than at present.

On the other hand, there is a tendency in this country whereby, in such matters as teaching and examination, the changes adopted are only gradually effected, and progress comes only by slow degrees. Accordingly, the general recommendations submitted in this report are such that they can be introduced easily and without any great alteration of the best present practice. It is the hope of the Committee that the recommendations, if adopted, will constitute merely the first stage in a gradual improvement both of teaching and of examinations. For the most part only broad lines of change are suggested: this has been done in order to leave as much freedom as possible to teachers for the development of their methods in the light of their experience.

*Is Uniformity imperative?*

The Committee do not consider that a single method of teaching mathematics should be imposed uniformly upon all classes of students; for the only variations then possible would be limited by the individuality of the teacher. In their opinion, different methods may be adopted for various classes of students, according to the needs of the students; and corresponding types of examination should be used.

It is generally, if not universally, conceded that a proper training in mathematics is an important part of a liberal education. The value of the training depends upon the comprehension of the aims of the mathematical subjects chosen, upon the grasp of the fundamental notions involved, and upon the attention paid to the logical sequence of the arguments. On the other hand, it is freely claimed that, in the training of students for technical aims such as the profession of engineering, a knowledge of results and a facility in using them are more important than familiarity with the mathematical processes by which the results are established with rigid precision. This divergence of needs belongs, however, to a later stage in the training of students. In the earliest stages, when the elements of mathematics are being acquired, the processes adopted can be substantially the same for all students; and many of the following recommendations are directed towards the improvement of those processes.

*Teaching of Practical Geometry.*

The former Committee recommended (and the present Committee desire to emphasise the recommendation) that the teaching of demonstrative geometry should be preceded by the teaching of practical and experimental geometry, together with a considerable amount of accurate drawing and measurement. This practice should be adopted, whether Euclid be retained, or be replaced by some authorised text-book or syllabus, or if no authority for demonstrative geometry be retained.

Simple instruments and experimental methods should be employed exclusively in the earliest stages, until the learner has become familiarised with some of the notions of geometry and

some of the properties of geometrical figures, plane and solid. Easy deductive reasoning should be introduced as soon as possible; and thereafter the two processes should be employed side by side, because practical geometry can be made an illuminating and interesting supplement to the reasoned results obtained in demonstrative geometry. It is desirable that the range of the practical course and the experimental methods adopted should be left in large measure to the judgment of the teacher; and two schedules of suggestions, intended for different classes of students, have been submitted to the Committee by Mr. Eggar and Professor Perry respectively, and are added as an Appendix to this Report.

### *Should there be a Single Authority in Geometry?*

In the opinion of the Committee, it is not necessary that one (and only one) text-book should be placed in the position of authority in demonstrative geometry; nor is it necessary that there should be only a single syllabus in control of all examinations. Each large examining body might propound its own syllabus, in the construction of which regard would be paid to the average requirements of the examinees.

Thus an examining body might retain Euclid, to the extent of requiring his logical order. But when the retention of that order is enforced, it is undesirable that Euclid's method of treatment should always be adopted; thus the use of hypothetical constructions should be permitted. It is equally undesirable to insist upon Euclid's order in the subject-matter; thus a large part of the contents of Books III. and IV. could be studied before the student comes to the consideration of the greater part of Book II.

In every case, the details of any syllabus should not be made too precise. It is preferable to leave as much freedom as possible, consistently with the range to be covered; for in that way the individuality of the teacher can have its most useful scope. It is the competent teacher, not the examining body, who best can find out what sequence is most suited educationally to the particular class that has to be taught.

A suggestion has been made that some Central Board might

be instituted to exercise control over the modifications made from time to time in every syllabus issued by an examining body. It is not inconceivable that such a Board might prove useful in helping to avoid the logical chaos occasionally characteristic of the subject known as Geometrical Conics. But there is reason to doubt whether the authority of any such Central Board would be generally recognised.

Opinions differ as to whether arithmetical notions should be introduced into demonstrative geometry, and whether algebraic methods should be used as substitutes for some of the cumbrous formal proofs of propositions such as those in Euclid's Second Book : for opinions differ as to the value of strictly demonstrative geometry, both for training and for knowledge. Those teachers who do not regard algebraic methods as proper substitutes for geometrical methods might still use them, as well as arithmetical notions, for the purpose of illustrating a proposition or explaining its wider significance. It is the general opinion of the Committee that some association of arithmetic and algebra with geometry is desirable in all cases where this may be found possible ; the extent to which it may be practised will depend largely upon the individual temperament of the teacher.

Every method of teaching demonstrative geometry has to face the difficulties inevitably associated with any complete and rigorous theory of proportion. In the opinion of the Committee, not merely is Euclid's doctrine of proportion unsuited for inclusion in elementary work, but it belongs to the class of what may be called university subjects. The Committee consider that the notion of proportion to be adopted in a school course should be based upon a combination of algebraical processes with the methods of practical geometry.

## *Examinations in Geometry.*

As regards examinations in geometry, the Committee consider that substantial changes in much of the present practice are desirable. In most, if not in all, of the branches of mathematics, and especially in geometry, the examination ought to be arranged so that no candidate should be allowed to pass unless he gives

evidence of some power to deal with questions not included in the text-book adopted. Such questions might comprise riders of the customary type arithmetical and algebraical illustrations and verifications, and practical examples in accurate drawing and measurement. The Committee consider the latter of particular importance when the range is of an elementary character; some influence will be exercised upon the teaching, and some recognition will be given to the course of practical geometry that should be pursued in the earlier stages.

### *Arithmetic and Algebra.*

The Committee are of opinion that, in the processes and explanations belonging to the early stages of these subjects, constant appeal should be made to concrete illustrations.

In regard to arithmetic, the Committee desire to point out, what has been pointed out so often before, that, if the decimal system of weights and measures were adopted in this country, a vast amount of what is now the subject-matter of teaching and of examination could be omitted as being then useless for any purpose. The economy in time, and the advantage in point of simplification, would be of the greatest importance. But such a change does not seem likely to be adopted at present; and the Committee confine themselves to making certain suggestions affecting the present practice. They desire, however, to urge that teachers and examiners alike should deal with only those tables of weights and measures which are the simplest and of most frequent practical use.

In formal arithmetic, the elaborate manipulation of vulgar fractions should be avoided, both in teaching and in examinations; too many of the questions that appear in examination papers are tests rather of mechanical facility than of clear thinking or of knowledge. The ideas of ratio and proportion should be developed concurrently with the use of vulgar fractions. Decimals should be introduced at an early stage, soon after the notion of fractions has been grasped. Methods of calculation, accurate only to specified significant figures, and, in particular, the practice of contracted methods, should be encouraged. The use of tables of simple functions should be begun as soon as

the student is capable of understanding the general nature of the functions tabulated; for example, the use of logarithms in numerical calculation may be begun as soon as the fundamental law of indices is known.

In regard to the early stages of algebra, the modifications (both in teaching and in the examinations) which are deemed desirable by the Committee are of a general character.

At first, the formulæ should be built on a purely arithmetical foundation, and their significance would often be exhibited by showing how they include whole classes of arithmetical results. Throughout the early stages, formulæ and results should frequently be tested by arithmetical applications. The arithmetical basis of algebra could be illustrated for beginners by the frequent use of graphs; and the practice of graphical processes in such cases can give a significance to algebraical formulæ that would not otherwise be obtained easily in early stages of the subject.

In passing to new ideas, only the simplest instances should be used at first, frequent reference still being made to arithmetical illustrations. Advance should be made by means of essential development, avoiding the useless complications of merely formal difficulties which serve no other purpose than that of puzzling candidates in examinations. Many of the artificial combinations of difficulties could be omitted entirely; the discussion of such as may be necessary should be postponed from the earlier stages. Teachers and examiners alike should avoid matters such as curious combinations of brackets; extravagantly complicated algebraic expressions, particularly fractions; resolutions of elaborate expressions into factors; artificially difficult combinations of indices; ingeniously manipulated equations: and the like. They have no intrinsic value or importance; it is only the mutual rivalry between some writers of text-books and some examiners that is responsible for the consideration which has been conceded to such topics.

## *General Remarks.*

If general simplification either on these or on similar lines be adopted, particularly if graphical methods are freely used, it will

be found possible to introduce, quite naturally and much earlier than is now the case, some of the leading ideas in a few subjects that usually are regarded as more advanced. Thus the foundations of trigonometry can be laid in connection with the practical geometry of the subject-matter of the Sixth Book of Euclid. The general idea of co-ordinate geometry can be made familiar by the use of graphs; and many of the notions underlying the methods of the infinitesimal calculus can similarly be given to comparatively youthful students long before the formal study of the calculus is begun.

*March* 13, 1902.

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## APPENDIX.

*Two Suggested Schedules of Experimental Geometry.*

(*Scheme submitted by* MR. EGGAR, *chiefly Geometrical, on Euclidean Lines.*)

Accurate measurements of lines, angles, areas, and (if possible) volumes, should precede any formal definitions. The following suggestions are intended for the earliest stages.

*Instruments.*—Hard pencil, compasses, dividers, straight edge graduated in inches and tenths, and in centimetres and millimetres; protractor (if rectangular, its connection with the division of the circle should be carefully pointed out); set-squares (45° and 60°); notebook of squared paper; tracing paper; scissors and loose paper for cutting out and folding.

It is important that careful draughtmanship and the use of properly adjusted instruments should be insisted on. All constructions should be drawn in fine pencil lines. Inaccurate work, or work done with soft or blunt pencils, should receive very little credit.

*Processes.*—Test of a straight line; intersection of two lines; notion (not definition) of a point; measurement of a length; estimation of the second place of decimals of inches or centimetres; use of set-squares for drawing parallel lines; construction and measurement of angles from 0° to 360° by the use of a

protractor; limits of error in setting off angles; test of a right angle; test for accuracy of set-squares: their use in drawing perpendiculars.

The drawing of parallels and perpendiculars by the aid of compasses; the bisection of angles and straight lines; construction of triangles from given dimensions; the fundamental properties of triangles verified and illustrated by drawing; similar triangles; the division of lines into equal parts and into parts in given proportion; test of equality of angles by the superposition of the angles of similar (not equal) triangles by means of tracing paper.

The construction of rectangles, parallelograms, and quadrilaterals, from adequate data; notion of a tangent line; construction of tangents to circles, using drawing-office methods; notion of a locus; construction of circles satisfying given conditions; verification of the properties of circles.

Measurement of area; use of squared paper; area of an irregular figure found by counting the number of squares.

Illustrations of propositions relating to the areas of squares, rectangles, parallelograms, and triangles. Calculation of these areas from given dimensions (*e.g.*, base and altitude), and verification by squared paper.

The length of the circumference of a circle determined experimentally (*e.g.*, by rolling a coin with an ink mark on its rim down an inclined sheet of paper, or by wrapping a strip of paper tightly round a cylinder, pricking the paper where it overlaps, unwrapping and measuring the distance between the two marks); the area of a circle determined by squared paper.

The area of a rectangular sheet of paper can be calculated from measurements in inches and in centimetres, and hence the number of square centimetres in a square inch can be obtained by division. To how many places of decimals may the result be regarded as accurate?

Construction of paper models of solids to illustrate the motions of surface and volume.

Measurement of volume should be illustrated by cubical bricks. Cubes of one inch and one centimetre can be obtained cheaply. Volumes of rectangular solids, prisms, cylinders, and cones, should be measured where possible, and the results verified by displacement of water if access to a physics laboratory is to be

had. Measurements of area and volume form a useful introduction to the notion of an algebraic formula.

As a pupil advances in elementary algebra, geometrical illustrations may be employed with advantage, *e.g.*, the verification with squared paper of the formulæ corresponding to the propositions of Euclid, Book II., graphs, the solution of quadratic equations with ruler and compasses.

(*Scheme submitted by* PROFESSOR PERRY: *this Scheme is intended to accompany a Course of Arithmetic, Algebra, and Experimental Science.*)

Practice in decimals, using scales for measuring such distances as 3·22 inches, or 12·5 centimetres.

Contracted and approximate methods of multiplying and dividing numbers; using rough checks in arithmetical work; evaluating formulæ.

*Mensuration.*—Testing experimentally the rule for the length of the circumference of a circle, using strings or a tape measure round cylinders, or by rolling a disc or sphere, or in other ways; inventing methods of measuring approximately the lengths of curves; testing the rules for the areas of a triangle, rectangle, parallelogram, circle, ellipse, surface of cylinder, surface of cone, &c., using scales and squared paper; propositions in Euclid relating to areas tested by squared paper, also by arithmetical work on actual measurements; the determination of the areas of an irregular plane figure (1) by using Simpson's or other well-known rules for the case where a number of equidistant ordinates or widths are given; (2) by the use of squared paper when equidistant ordinates are not given, finding such ordinates; (3) weighing a piece of cardboard and comparing with the weight of a square piece; (4) counting squares on squared paper to verify rules. Rules for volumes of prisms, cylinders, cones, spheres, and rings, verified by actual experiment; for example, by filling vessels with water, or by weighing objects of these shapes made of material of known density, or by allowing such objects to cause water to overflow from a vessel.

The determination of the volume of an irregular solid by each of the three methods for an irregular area, the process being first to

obtain an irregular plane figure in which the varying ordinates or widths represent the varying cross-sections of the solid; volumes of frustra of pyramids and cones; computation of weights from volumes when densities are given.

Stating a mensuration rule as an algebraic formula. In such a formula any one of the quantities may be the unknown one, the others being known. Numerical exercises in mensuration. The experimental work in this subject ought to be taken up in connection with practice in weighing and measuring generally, finding specific gravities, illustrations of the principle of Archimedes, the displacement of floating bodies, and other elementary scientific work. A good teacher will not overdo this experimental work: he will preserve a proper balance between experimental work, didactic teaching, and numerical exercise work.

*Use of squared paper.*—The use of squared paper by merchants and others to show at a glance the rise and fall of prices, of temperature, of the tide, &c. The use of squared paper should be illustrated by the working of many kinds of exercises, but it should be pointed out that there is a general idea underlying them all. The following may be mentioned:—

Plotting of statistics of any kind whatsoever of general or special interest; what such curves teach; rates of increase.

Interpolation, or the finding of probable intermediate values; probable errors of observation; forming complete price lists by manufacturers; finding an average value; areas and volumes as explained above.

The plotting of simple graphs; determination of maximum and minimum values; the solution of equations. Very clear notions of what we mean by the roots of equations may be obtained by the use of squared paper.

Determination of laws which exist between observed quantities, especially of linear laws.

Corrections for errors of observation when the plotted quantities are the results of experiment.

*Geometry.*—A knowledge of the properties of straight lines, parallel lines, right angles, and angles of 30°, 45°, and 60°, obtained by using and testing straight-edges and squares; dividing lines into parts in given proportions, and other experimental illustrations of the Sixth Book of Euclid; the definitions of the sine, cosine,

and tangent of an angle, and the determination of their values by graphical methods; setting out of angles by means of a protractor, when they are given in degrees or radians, also (for acute angles) by construction when the value of the sine, cosine, or tangent is given; use of tables of sines, cosines, and tangents; the solution of a right-angled triangle by calculation and by drawing to scale; the construction of any triangle from given data; determination of the area of a triangle. The more important propositions of Euclid may be illustrated by actual drawing. If the proposition is about angles, these may be measured in degrees by means of a protractor, or by the use of a table of chords; if it refers to the equality of lines, areas, or ratios, lengths may be measured by a decimal scale, and the necessary calculations made arithmetically. This combination of drawing and arithmetical calculation may be freely used to illustrate the truth of a proposition. A good teacher will occasionally introduce demonstrative proof as well as mere measurement.

Defining the position of a point in space by its distances from three co-ordinate planes. What is meant by the projection of a point, line, or a plane figure, on a plane? Simple models may be constructed by the student to illustrate the projections of points, lines, and planes.

The distinction between a scalar quantity and a vector quantity; addition and subtraction of vectors; experimental illustrations, such as the verification by the student himself of the triangle and polygon of forces, using strings, pulleys, and weights.

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